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CONTRIBUTION TO THE THEORY OF HEAT TRANSFER AT STRONG TEMPERATURE AND PRESSURE GRADIENTS IN THE DIRECTION OF THE FLOW, FOR PLANE ROTATIONALLY SYMMETRICAL BOUNDARY LAYERS

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CONTRIBUTION TO THE THEORY OF HEAT TRANSFER AT STRONG TEMPERATURE AND PRESSURE GRADIENTS IN THE DIRECTION OF THE FLOW, FOR PLANE ROTATIONALLY SYMMETRICAL BOUNDARY LAYERS **/1

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Analysis showing the Nusselt classical theory and classical methods cannot be applied to the solution of heat-transfer problems in high-speed aerodynamics, because they neglect the effect of temperature gradients in the direction of flow. For laminar boundary layers, this effect can be substantial. An approximate theory is developed which takes into account the effect of such temperature gradients for any temperature distribution at the wall and which is applicable to both incompressible and compressible turbulent boundary layers. For similar boundary-layer solutions, the theory demonstrates (in analytical form) that the effect of heat transfer of a pressure gradient in the direction of flow can also be appreciable. A distinctive feature of the proposed theory is that the temperature and velocity profiles of the boundary layer need not be expressed in series form. The only unknowns besides the boundary-layer thickness are a form parameter H(x) for the velocity profile and a "modification parameter" K(x) for the temperature profile. These three unknowns can be determined from a simultaneous system of three first-

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** Numbers in the margin indicate pagination in the original foreign text.

order differential equations. The theory is applied to several examples. Author

1. Preliminary Remarks

The problem of heat transfer between solid walls and flowing media touches upon two large disciplines, namely, thermodynamics and fluid mechanics. For both disciplines, this type of heat transfer had for long been a theoretically extremely difficult "marginal" problem, whose solution had of necessity been left to purely empirical means.

Only after development of the boundary-layer theory by L.Prandtl (Bibl.1) and the similitude theory of heat transfer by W.Nusselt (Bibl.2), at the beginning of this Century, did strict solutions or (in the case of turbulent boundary layers) satisfactory estimates of this problem become possible, in some important special cases. The well-known theoretical and semi-empirical laws which, in the case of forced convection, express the Nusselt "heat transfer coefficient" Nu as a function of the Prandtl number Pr (= ratio of impulse transfer to heat exchange) and of the Reynolds number Re (= ratio of inertia force to viscosity force), were important results of this work.

Recently, the problematics of heat transfer was further complicated: In rapidly flowing media, energy conversions (generation of heats of compression and friction) take place which - over the variation of physical constants with temperature and pressure - result in a strong interaction of the flow and temperature boundary layers. The laws of conservation of momentum and energy which were independently solvable when treating the heat transfer problem on the basis of Nusselt's similitude theory with constant physical constants, /2
can be solved in most modern problems only as coupled systems of partial dif-

ferential equations. In addition, the boundary conditions are frequently much more complex because of certain effects, such as strong wall-temperature and pressure gradients in the direction of flow.

Due to the development of efficient mathematical and computational (specifically electronic) means, new possibilities were created for an exact or approximate solution of this difficult problem complex of heat transfer.

Modern mathematical aids include methods for an approximate solution of partial differential equations by averaging the equation result in one coordinate direction (by forming so-called integral conditions as ordinary differential equations for free parameters of a solution argument); these are methods related to Galerkin's method (Bibl.3), familiar to mathematicians.

In this paper, after a brief review of the classical theory of flow and temperature boundary layers, we will demonstrate that the use of suitable solution arguments and the method of integral conditions eliminates the need for excessive simplifications, which were required for most of the conventional solutions. This will furnish an answer to some questions on heat transfer in technically important problems, which (so far as known to the author) have never been solved.

2. Problem Formulation /3

One of the basic simplifications of most known theories (for example) is the assumption that the temperature T_w of the wall, along which the heat transfer takes place, remains constant in the direction of flow x $\left(\frac{dT_w}{dx} = 0\right)$ and that no pressure gradient exists in the direction of flow $\left(\frac{dp}{dx} = 0\right)$; see, for example, L.Crocco (Bibl.4) and E.R.van Driest (Bibl.5).

D.R.Chapman and M.W.Rubesin (Bibl.6) in 1949 as well as H.Schlichting

(Bibl.7) in 1951 were able to demonstrate, in the case of laminar incompressible and compressible boundary layers, that at least the influence of the variable wall temperature on the heat transfer may be considerable.

In a frequently quoted report on similar solutions of the compressible laminar boundary layer with heat transfer, T.V.Li and H.T.Nagamatsu (Bibl.8) showed that a pressure gradient, in the direction of flow, will have an - although moderate - influence on the solution.

In the theory to be developed here, the influence of the gradients $\frac{dT_w}{dx}$ and $\frac{dp}{dx}$ is to be determinable for compressible laminar and turbulent boundary layers.

This theory also covers the special case of heat transfer for very thin threads exposed to longitudinal flow, in a rather simple manner; this case is of considerable importance in the technique of the filament production from thermoplastic materials in blast jets.

The theory itself, however, will be restricted in the conventional manner to the case of stationary flow at high Reynolds numbers, in the sense of Prandtl's boundary-layer theory. Emphasis will be on the case of ideal gases, with the usual value of unity given to the Prandtl number (Pr = 1), as in all other theories. In this case, the equation of state for ideal gases

$$\rho = \frac{p}{g\Re T} \qquad \begin{pmatrix} \rho = \text{density} \\ p = \text{static pressure} \\ \Re = \text{gas constant} \\ g = \text{acceleration of gravity} \end{pmatrix} \tag{1}$$

as well as Sutherland's law for the temperature dependence of the molecular viscosity

$$\mu(T) = 1.486 \cdot 10^{-7} \frac{T^{3/2}}{T+C}, \begin{pmatrix} T \text{ in } {}^{\circ}\text{KELVIN}, \\ \mu \text{ in } \text{kp sec/m}^2 \end{pmatrix} \tag{2}$$

at C = 110.6 for air will be valid. The specific heats c_p and c_v as well as their ratio $\varkappa = \frac{c_p}{c_v}$ will be assumed as constant. Finally, only the practically most important case of heat transfer by forced convection will be treated here, in which the buoyancy forces, due to density differences, can be neglected with respect to inertia, pressure, and viscosity forces.

In the case of turbulent flow, only the time-average values of the velocities, pressures, temperatures, and physical constants are of interest for our particular problem complex. The momentum and heat exchange, increased by the turbulent motion, is taken into consideration by "effective" values for the viscosity and the heat transfer coefficient μ_e and λ_e, as the sum of the molecular and "apparent" quantities:

$$\boxed{\mu_e = \mu(T) + \mu_s \; ; \; \lambda_e = \lambda(T) + \lambda_s} \qquad (2a)\ (2b)$$

The apparent values μ_s and λ_s are obtained from well-known empirical laws [see, for example, H.Schlichting (Bibl.9)].

The Prandtl number Pr, in the case of a turbulent boundary layer, must be formed with the effective quantities μ_e and λ_e:

$$Pr_t = \frac{\mu_e c_p}{\lambda_e} \qquad (2c)$$

3. Physical Principles /5

Let us assume a two-dimensional flow along a fixed boundary. The x-axis of the Cartesian coordinate system is assumed to coincide with the (only slightly curved) boundary. The amount of heat q(x) (energy per unit time and unit area), transferred at a fixed point x in the y-direction perpendicular to the wall, will then, because of the viscosity condition of the flow medium along the boundary, be completely given by the Fourier heat conduction formula

$$q_w(x) = -\left(\lambda \frac{\partial T}{\partial y}\right)_{y=0} , \tag{3}$$

even in the case of a turbulent boundary layer. Equation (3) defines the molecular "heat transfer coefficient"

$$\lambda = \lambda(T) \tag{4}$$

of the flowing medium, as a known physical constant which depends only on the temperature T.

The problem of heat transfer between a stationary wall and a flowing medium thus is solved for laminar or turbulent boundary layers, at the general problematics and boundary conditions of Section 2, if the temperature gradient $\left(\frac{\partial T}{\partial y}\right)_{y=0}$ along the wall is known*.

Thus, the temperature field T(x, y), i.e., under our assumption of high Reynolds number, the temperature boundary layer, must be determined in its interaction with the flow boundary layer. In addition to the influence of forced convection and heat conduction, the temperature variations due to compression and friction must be considered in full.

The laws of conservation of mass and momentum have the usual form in our problem formulation. It only must be taken into consideration that the physical constants are variable. In formulating the law of conservation of energy, it is preferable to use as basis the concept of total energy h, as the sum of /6
enthalpy and kinetic energy (all energies referring to unit mass):

$$h = i + \frac{u^2}{2} + \frac{v^2}{2} \approx i + \frac{u^2}{2} \quad \left(\text{because of } \frac{v}{u} \ll 1 \text{ in the boundary layer}\right) \tag{5}$$

* An additional heat transport by radiation at high temperatures can be readily calculated in accordance with conventional laws [see, for example (Bibl.12)].

In an isentropic flow, i.e., outside of the flow and temperature boundary layer, the total energy h is known to be constant. Within the boundary layer, this total energy can be changed only by heat supply or heat dissipation due to heat conduction or by the work of shear stresses. The work of pressures (technical work as well as compression or expansion work) does not explicitly enter this energy balance since, on introduction of the total energy, the interaction between kinetic energy and enthalpy is implictly covered.

4. System of Equations

For the assumed stationary two-dimensional boundary-layer flow (at large Reynolds number), the laws of the conservation of mass, momentum, and energy have the following form:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad \underline{\text{Mass}} \tag{6}$$

$$\frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} = -\frac{dp}{dx} + \frac{\partial \tau}{\partial y} \qquad \underline{\text{Momentum}} \tag{7}$$

$$\frac{\partial(\rho u h)}{\partial x} + \frac{\partial(\rho v h)}{\partial y} = \frac{\partial}{\partial x}\left(\lambda_e \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_e \frac{\partial T}{\partial y}\right) + \frac{\partial(u\tau)}{\partial y} \qquad \underline{\text{Energy}} \tag{8}$$

$$\tau = \mu_e \frac{\partial u}{\partial y} \qquad \underline{\text{Effective shear stress}} \tag{8a}$$

with the supplementary relations (1) - (4) for ρ, μ_e, and λ_e. For the six unknowns u, v, h, ρ, μ_e, and λ_e, a total of six equations are thus available.

On multiplying eq.(7) by the velocity components u, an equation for the mechanical energies is obtained:

$$\frac{\partial(\rho u \cdot \frac{u^2}{2})}{\partial x} + \frac{\partial(\rho v \cdot \frac{u^2}{2})}{\partial y} = -u\frac{dp}{dx} + u\frac{\partial \tau}{\partial y}. \tag{9}$$

Subtracting eq.(9) from eq.(8) and taking eq.(5) into consideration will yield the well-known equation for the enthalpy boundary layer

$$\frac{\partial(\rho u i)}{\partial x} + \frac{\partial(\rho v i)}{\partial y} = u\frac{dp}{dx} + \tau\frac{\partial u}{\partial y} + \frac{\partial}{\partial x}\left(\lambda_c\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_c\frac{\partial T}{\partial y}\right). \tag{10}$$

For the case of ideal gases, with

$$i = c_p T \tag{11}$$

eq.(10) is the equation for the temperature field.

The boundary conditions for eqs.(6) - (10) are as follows:

$$\begin{aligned} y = 0 &: \quad u = v = 0 \;;\quad T = T_w(x) \\ y \to \delta &: \quad u \to u_\delta \;;\quad T \to T_\delta . \end{aligned} \tag{12}$$

A prescribable boundary condition (problem formulation) is the pressure p(x) = = $p_\delta(x)$ impressed on the boundary layer and connected with the velocity $u_\delta(x)$ along the edge of the boundary layer by the Bernoulli equation

$$\int\frac{dp}{\rho} + \frac{u_\delta^2}{2} = \text{const} \;;\quad \frac{dp}{dx} = -\rho_\delta u_\delta\frac{du_\delta}{dx}. \tag{13, 14}$$

The asymptotic transition from u to u_δ and from T to T_δ, at the assumed large Reynolds numbers, takes place practically within a boundary-layer region whose approximate thickness in the case of the flow boundary layer is denoted by δ_s and in the case of the temperature boundary layer by δ_T. In the case of a laminar boundary layer, according to Prandtl's approximation rule (Bibl.1), /8 the following applies in first approximation:

$$\frac{\delta_s}{L} \approx c\,\frac{\sqrt{x/L}}{\sqrt{R_L}} \quad ; \quad \frac{\delta_T}{L} \approx c\,\frac{\sqrt{x/L}}{\sqrt{P_r R_L}} \qquad (15)\ (16)$$

with

$$R_L = \frac{\rho_s u_s L}{\mu_s} \quad ; \quad P_r = \frac{\mu c_p}{\lambda} \quad ; \quad c \approx 5 \qquad (17)\ (17a)\ (17b)$$

On the basis of eqs.(15) and (16), this yields

$$\frac{\delta_s}{\delta_T} \approx \sqrt{P_r} \qquad (18)$$

Thus, we have $\delta_s = \delta_T$ for Pr = 1. The flow and temperature boundary layers in this case which is approximately given for gases (for example, air at Pr = = 0.72) and for a turbulent boundary layer, have practically equal thickness. For Pr > 1, i.e., in the case of laminar flow of fluids (water: Pr ≈ 7 at 20° C; oil: Pr ≈ 1600 at 20° C), the temperature boundary layer - according to eq.(18) - is much thinner than the flow boundary layer.

The thicknesses δ_s and δ_T are used only for estimates. In an exact solution of the system (6), (7), (8), these quantities do not enter.

The estimating formulas (15) and (16) for δ_s and δ_T will be used by us for simplifying eqs.(8) and (10). Since the wall distances, within the temperature boundary layer, are of the order of magnitude of $y \sim \delta_T \sim \frac{L}{\sqrt{PrR_L}}$, the following is valid (using $\frac{x}{L} \sim 1$) for the order of magnitude of zero of the heat conduction terms in eqs.(8) and (10):

$$O\left[\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)\right] = \left(\frac{\delta_T}{L}\right)^2 \cdot O\left[\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)\right] \qquad (19)$$

If we let $\frac{dT_w}{dx}$ and thus also $\frac{\partial T}{\partial x}$, increase up to an order of magnitude of

$$\frac{\partial T}{\partial y} \sim \frac{T_w - T_\delta}{\delta_T} \tag{20}$$

then $\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right)$ will be smaller than $\frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right)$ by a factor of $\frac{\delta_T}{L} \sim$ $\sim \frac{1}{\sqrt{Pr R_L}}$.

Thus, using the restriction of $\frac{dT_w}{dx}$ to

$$\frac{dT_w}{dx} \sim \frac{\partial T}{\partial y} \lesssim \frac{T_w - T_\delta}{\delta_T} \tag{21}$$

the term $\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right)$ in eqs.(8) and (10) can be neglected for $\frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right)$. In each individual case, it must be checked whether the condition (21), which excludes extreme temperature gradients $\frac{dT_w}{dx}$, is satisfied.

With this simplification which, in an analogous manner, can be used also for the turbulent boundary layer, eqs.(8) and (10) can finally be written as follows [taking eqs.(5) and (6) into consideration]:

$$\begin{aligned} \rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} &= \frac{\partial}{\partial y}\left(\lambda_e \frac{\partial T}{\partial y}\right) + \frac{\partial (u\tau)}{\partial y} \\ &= \frac{1}{Pr}\frac{\partial}{\partial y}\left(\mu_e \frac{\partial h}{\partial y}\right) + \frac{Pr-1}{Pr}\frac{\partial (u\tau)}{\partial y} \end{aligned} \tag{22}$$

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$$\boxed{\rho u \frac{\partial i}{\partial x} + \rho v \frac{\partial i}{\partial y} = u \frac{dp}{dx} + \tau \frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left(\lambda_e \frac{\partial T}{\partial y}\right)} \tag{23}$$

For Pr = 1, eq.(22) is further simplified to

$$\boxed{\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial h}{\partial y}\right)} , \tag{24}$$

but remains valid for arbitrary compressible external flows (for arbitrary pressure gradients $\frac{dp}{dx}$) with or without heat transfer, possibly also for turbulent boundary layers.

5. Exact Solutions for the Temperature Boundary Layer

5.1 Pr = 1, Heat-Insulated Wall, $\frac{dp}{dx}$ Arbitrary

In this case,

$$h = \text{const} = h_\delta \tag{25}$$

is a particular integral of the equation (24), linear in h, so that the total energy h is constant also within the boundary layer despite the created heat of friction, at an arbitrary pressure gradient $\frac{dp}{dx}$.

From the definition equations (5) and (11) for h and i, the "temperature profile"

$$\frac{T(x,y)}{T_\delta(x)} = 1 + \frac{u_\delta^2/2}{c_p T_\delta}\left[1 - \left(\frac{u}{u_\delta}\right)^2\right] \tag{26}$$

is immediately obtained for gases. If, using the well-known gas-dynamic relation

$$\frac{u_\delta^2/2}{c_p T_\delta} = \frac{\kappa - 1}{2} M_\delta^2 = \frac{\text{kinetic energy}}{\text{enthalpy}}, \tag{27}$$

the local Mach number /11

$$M_\delta(x) = \frac{u_\delta(x)}{a_\delta(x)} \tag{28}$$

is introduced, the connectivity law between $\frac{T}{T\delta}$ and $\frac{u}{u\delta}$

$$\boxed{\frac{T(x,y)}{T_\delta(x)} = \frac{T}{T_\delta}\left(\frac{u}{u_\delta}, M_\delta\right) = 1 + \frac{\kappa - 1}{2} M_\delta^2\left[1 - \left(\frac{u}{u_\delta}\right)^2\right]} \tag{29}$$

is obtained for eq.(26). Along the wall y = 0, u = 0, the "characteristic temperature" T_e prevails. In our case, this temperature is equal to the adiabatic stagnation temperature T_0

$$\boxed{\left(\frac{T_e}{T_\delta}\right)_{P_r=1} = \frac{T_0}{T_\delta} = 1 + \frac{\kappa-1}{2} M_\delta^2 .} \tag{30}$$

On the basis of eq.(29), the flow boundary layer can be calculated according to known methods [see for example (Bibl.9) p.318].

5.2 <u>Pr Arbitrary, $\frac{dp}{dx} = 0$, $\frac{\partial i}{\partial x} = c_p \frac{\partial T}{\partial x} = 0$, $\left(\frac{dT_w}{dx} = 0,\ T_w = \text{const}\right)$;</u>

<u>Heat Transfer Arbitrary; $\left(\frac{\partial T}{\partial y}\right)_{y=0}$ Arbitrary</u>

According to L.Crocco (Bibl.4) and E.R.van Driest (Bibl.5), the quantities x and y in eqs.(7) and (23) are replaced by the independent variables x and u. Using the dimensionless shear stress

$$\tau^* = 2 \frac{\tau}{\rho_\delta u_\delta^2} \sqrt{R_x} \qquad \left(R_x = \frac{\rho_\delta u_\delta x}{\mu_\delta} \right) \tag{31}$$

and the dimensionless enthalpy

$$i^* = \frac{i}{i_\delta} , \tag{32}$$

eqs.(7) and (23) assume the form

$$\tau^* \tau^{*''} + \frac{\mu}{\mu_\delta} \frac{\rho}{\rho_\delta} \frac{u}{u_\delta} = 0 \tag{33}$$

$$\left(\frac{i^{*'}}{P_r}\right)' + (1 - P_r) \frac{\tau^{*'}}{\tau^*} \frac{i^{*'}}{P_r} - \frac{u_\delta^2}{i_\delta} = 0 . \tag{34}$$

where the prime indicates a differentiation to $\frac{u}{u_\delta}$. The boundary conditions /12 will then be

$$\left\{\begin{array}{llll} \frac{u}{u_\delta} = 0 & : \left(\frac{\partial \tau^*}{\partial y}\right)_{y=0} = \frac{dp}{dx} = 0 & ; & i^* = i^*_w \\ \frac{u}{u_\delta} = 1 & : \tau^* = \tau^*_\delta = 0 & ; & i^* = i^*_\delta = 1 \end{array}\right. \qquad (35)$$

For arbitrary but constant Prandtl numbers Pr and for a shear stress distribution $\frac{\tau^*}{\tau^*_w} = \frac{\tau}{\tau_w}$, assumed as known, eq.(34) which is linear in i can be integrated. The solution reads

$$\boxed{i^* = \frac{i}{i_\delta} = 1 + \frac{i_e - i_w}{i_\delta}(f_1 - 1) + r\frac{u_\delta^2/2}{i_\delta}(1 - f_2)} \qquad (36)$$

Here, f_1 and f_2 are certain integral expressions that can be evaluated, at known shear stress distribution, for arbitrary Prandtl numbers.

The following is valid:

$$f_1 = \frac{\Sigma}{s} \quad ; \quad f_2 = 2\frac{P}{r} \qquad (37)\ (38)$$

$$\Sigma = P_r \int_0^{u/u_\delta} \left(\frac{\tau}{\tau_w}\right)^{P_r - 1} d\left(\frac{u}{u_\delta}\right) \quad ; \quad P = P_r \int_0^{u/u_\delta} \left(\frac{\tau}{\tau_w}\right)^{P_r-1} \left(\int_0^{u/u_\delta} \left(\frac{\tau}{\tau_w}\right)^{-P_r} d\left(\frac{u}{u_\delta}\right)\right) d\left(\frac{u}{u_\delta}\right) \qquad (39)\ (40)$$

$$\left(\Sigma\right)_0^1 = s \qquad \text{= Reynolds analogy factor} \qquad (41)$$

$$\left(2P\right)_0^1 = r \qquad \text{= characteristic temperature coefficient (recovery factor)} \qquad (42)$$

$$i_e = i_\delta\left(1 + r\frac{u_\delta^2/2}{i_\delta}\right) \quad (43)\text{; for gases: } T_e = T_\delta\left(1 + r\frac{\kappa - 1}{2}M_\delta^2\right) \qquad (44)$$

Since $\frac{T}{T_w}$ is not known at first, the system (33) - (36) can be solved only /13 by iteration. However, eq.(36) indicates that the outer boundary $y = \delta_T$ $\left(\frac{i}{i_\delta} = \frac{T}{T_\delta} = 1\right)$ is located at a point where f_1 and f_2 practically reach a value of 1. In the special case of Pr = 1, the difficulty due to the unknown slope of $\frac{T}{T_w}$ $\left(\frac{u}{u_\delta}\right)$ is eliminated since the exponent Pr - 1 = 0 in eqs.(39) and (40) makes an integration of eq.(34) independent of the shear stress distribution. For Pr = = 1, this will yield

$$f_1 = \frac{u}{u_\delta} \quad ; \quad f_2 = \left(\frac{u}{u_\delta}\right)^2 \qquad (45)\ (46)$$

$$s = r = 1 \quad ; \quad i_e = i_0 \quad , T_e = T_0 ; \qquad (47)\ (48)$$

while the solution (36), at $i = c_p T$ has the form

$$\frac{T(x,y)}{T_\delta(x)} = \frac{T}{T_\delta}\left(\frac{u}{u_\delta}, M_\delta, \frac{T_0 - T_w}{T_\delta}\right)$$
$$= 1 + \frac{T_0 - T_w}{T_\delta}\left(\frac{u}{u_\delta} - 1\right) + \frac{\kappa - 1}{2} M_\delta^2 \left[1 - \left(\frac{u}{u_\delta}\right)^2\right]. \qquad (49)$$

Consequently, the flow and temperature boundary layers are intimately linked also in this case. In this connectivity law, not only the Mach number M_δ but also the parameter $\left(\frac{T_0 - T_w}{T_\delta}\right)$ plays an important role for the heat transfer.

According to eq.(5) and at Pr = 1, the total energy h will then read

$$\frac{h - h_\delta}{h_w - h_\delta} = 1 - \frac{u}{u_\delta}. \qquad (50)$$

For an incompressible flow, the kinetic energy $\frac{u^2}{2}$ in h can be neglected for the enthalpy $c_p T$ so that $h_{M_\delta \to 0} \to c_p T$ is valid. Then, eq.(50) for $M_\delta \to 0$, $T_0 \to$ $\to T_\delta$ will read as follows:

$$\boxed{\frac{T-T_w}{T_\delta-T_w} = \frac{u}{u_\delta}}. \qquad (51)$$

Consequently, the boundary layers of both temperature and flow are congruent in this case. /14

If the dependence of the functions f_1 and f_2 in eqs.(36) on the Prandtl number Pr, at given shear stress distribution $\frac{\tau}{\tau_w}$, is investigated, for example, for the shear stress distribution of the laminar boundary layer over a plane plate according to H.Blasius (Bibl.10)*, the result plotted in Figs.1 and 2 will be obtained: Here, f_1 and f_2 will "practically" reach the asymptotic boundary value 1 at relatively small distances from the wall, which decrease further with increasing Prandtl number. This corresponds to the estimate given in eq.(16), according to which the thickness of the temperature boundary layer decreases with increasing Prandtl number**.

For the case of turbulent boundary layers and assuming a linear slope of the shear stress with distance from the wall, $\frac{\tau}{\tau_w} = 1 - \frac{y}{\delta_s}$ [see (Bibl.14)], a result agreeing with Figs.1 and 2 is obtained.

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* According to E.R. van Driest (Bibl.5), for a laminar boundary layer, the following expression is valid in satisfactory approximation within the Mach number range of $0 < M_\delta < 4$:

$$\frac{\tau}{\tau_w} \approx 1 - \left(\frac{u}{u_\delta}\right)^{3.5}$$

** Since Prandtl numbers above 1 generally are in question for liquids, i.e., for incompressible media, the term with the factor $\frac{\frac{u_\delta^2}{2}}{i_\delta}$ or $\frac{x-1}{2} M_\delta^2$ for gases vanishes in eq.(36) at Pr > 1. In this manner, the theory for Pr > 1 is again simplified. For molten metals, as another example of practically incompressible media with Prandtl numbers much smaller than 1 (mercury with Pr ≈ 0.023 at 20° C), corresponding simplifications of the theory are obtained.

5.3 Relations for the Heat Transfer in the Special Exact Solution according to Crocco and van Driest (Reynolds Analogy)

Using the relation (36), valid for arbitrary Prandtl numbers, eq.(3) will yield the following relation for the heat q_w transferred along the wall at $i = c_p T$, after an elementary intermediate calculation:

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$$q_w(x) = -\left(\lambda \frac{\partial T}{\partial y}\right)_{y=0} = -\left(\lambda \frac{dT}{du} \cdot \frac{\partial u}{\partial y}\right)_{y=0} = -\frac{1}{s} \frac{\tau_w}{\varrho_\delta u_\delta^2} \cdot \frac{T_e - T_w}{T_\delta} \cdot \frac{c_p T_\delta}{u_\delta^2} \varrho_\delta u_\delta^3 \qquad (52)$$

or, in dimensionless form, at

$$c_f = \frac{\tau_w}{\varrho_\delta u_\delta^2} \qquad (52a)$$

as local friction coefficient

$$\boxed{\frac{q_w}{\varrho_\delta u_\delta^3} = -\frac{1}{2} \frac{c_f}{s} \cdot \frac{T_e - T_w}{T_\delta} \cdot \frac{c_p T_\delta}{u_\delta^2/2} = -\frac{r}{2} \frac{c_f}{s} \Theta} \; . \qquad (53)$$

Here,

$$\Theta = \frac{(T_e - T_w)/T_\delta}{r \dfrac{u_\delta^2/2}{c_p T_\delta}} = \frac{T_e - T_w}{r \frac{\kappa - 1}{2} \cdot M_\delta^2 T_\delta} = \frac{T_e - T_w}{T_e - T_\delta} \qquad (54)$$

is a parameter for the heat transfer. The entire heat, exchanged along the wall, is obtained by integration of $q_w(x)$ in the x-direction. The symbol

$$\frac{c_f}{s} = ST \qquad (55)$$

is known also as Stanton's number which (within the validity range of the theory described here, i.e., for $\frac{dT_w}{dx} = 0$ and $\frac{dp}{dx} = 0$) is connected with the Nusselt number Nu by the relation

$$Nu = ST \cdot Pr \cdot R_L \; . \qquad (56)$$

The simple relation (53) for the heat transfer, known also as Reynolds analogy between friction (c_f) and heat transfer (q), is valid in the same manner for laminar and turbulent boundary layers.

However, this relation becomes uncertain as soon as the assumptions $\frac{dT_w}{dx} = 0$ and $\frac{dp}{dx} = 0$ are not satisfied. /16

No general exact solution of the system (6), (7), (22), or (23) that would cover the influence of $\frac{dT_w}{dx}$ $\left(\text{by } \frac{\partial T}{\partial x}\right)$ and $\frac{dp}{dx}$, is known. Below, we will derive an approximate solution that makes use of physical and mathematical basic principles, which can be successfully applied to the approximate solution of the problem of laminar and turbulent flow boundary layers.

6. Basic Principle of an Approximate Solution of General Problematics

In the field of laminar flow boundaries, basic properties of the solutions (of the velocity profiles) had been determined in special cases by exact solutions, such as "similar solutions" for accelerated and decelerated flows of the type of $u_\delta \sim x^m$ at m = const, in accordance with D.R.Hartree (Bibl.13). These orienting solutions furnished important criteria for the construction of solution arguments for velocity profiles, expected in the general problematics [at arbitrary slope of $u_\delta(x)$].

In these cases, arguments for the velocity profile, of the following form were obtained:

$$\frac{u(x,y)}{u_\delta(x)} = F\left(\frac{y}{\delta_s(x)}, H(x)\right), \tag{57}$$

where $\delta_s(x)$ denotes the boundary layer thickness and H(x) is a form factor of the two unknowns of the solution argument. For determining these unknowns, the integral conditions for momentum and energy, derived from eqs.(7) and (9) by partial integration to y, can be used. This approximate solution is then necessarily identical with the exact solution, for flows of the type $u_\delta \sim x^n$.

In a similar manner, we will use the exact solutions treated in Section 5 for constructing a solution argument which contains these exact solutions as special cases, but provides one more - or, if necessary, several more - free parameters that permit an adaptation to more general problem formulations. /17

Known solution arguments for the temperature profile $\frac{T}{T_\delta}$ use an expansion in powers of the running length $\frac{x}{L}$ with coefficient functions of the distance from the wall y [H.Schlichting (Bibl.9)] or in powers of the wall distance y with coefficients depending on x [D.N.Morris, J.W.Smith (Bibl.11)]. In view of the character of the exact solutions discussed in Section 5, it seems advisable to construct the solution argument for $\frac{T}{T_\delta}$ with the functions f_1 and f_2 of the exact solution (36) derived by van Driest (Bibl.5) or else, at Pr = 1, to use an expansion in powers of the velocity $\frac{u}{u_\delta}$ in which case, for physical reasons [because of eq.(49)], the terms with $\frac{u}{u_\delta}$ and $\left(\frac{u}{u_\delta}\right)^2$ might be sufficient.

The special solution (36), exactly valid for $\frac{dT_w}{dx} = 0$, $\frac{dp}{dx} = 0$, will be

$$\frac{T}{T_\delta} = a + b f_1 + c f_2 \tag{58}$$

with

$$a = 1 - \frac{T_e - T_w}{T_\delta} + r\frac{\varkappa - 1}{2} M_\delta^2; \quad b = \frac{T_e - T_w}{T_\delta}; \quad c = -r\frac{\varkappa - 1}{2} M_\delta^2 \qquad (59)\ (60)\ (61)$$

We modify this solution by introducing a form parameter K(x) independent of x, such that, for $f_1 = f_2 = 1$ (with a + b + c = 1), we have $\frac{T}{T_\delta} = 1$

$$\boxed{\frac{T}{T_\delta} = a + (b + K(x)) f_1 + (c - K(x)) f_2} \tag{62}$$

with a, b, c according to eqs.(59) - (61). For Pr = 1, eq.(62) is transformed into

$$\boxed{\frac{T}{T_\delta} = a + (b + K(x)) \frac{u}{u_\delta} + (c - K(x)) \left(\frac{u}{u_\delta}\right)^2} \tag{63}$$

The modified arguments (62) and (63) specifically have the property, required /18 by Schlichting (Bibl.7), that a heat transfer $\left[\left(\frac{\partial T}{\partial y}\right)_{y=0} \neq 0\right]$ is still possible at a temperature gradient in the direction of flow, despite b = 0, i.e., despite $T_w = T_e$ or, in the case of an incompressible flow, despite $T_w = T_\delta$. Beyond this, the solution argument for $\frac{T}{T_\delta}$, compared to the conventional arguments with expansions in series to x, has the advantage that $T_w(x)$ and $u_\delta(x)$ can be arbitrary functions of x and that this argument is directly valid for laminar and turbulent boundary layers.

The relation (53) generalized on the basis of the argument (62), for the heat transfer then reads

$$\frac{q_w}{\varrho_\delta u_\delta^3} = -\frac{r}{2} ST\,\Theta\left(1 + \frac{K(x)}{b}\right). \tag{64}$$

The proportionality between friction (c_f resp. ST) and heat flux (q) thus is retained in this generalized theory despite the fact that, strictly, it can no longer be valid.

The unknown K(x) of the approximation theory, added to the existing $\delta_s(x)$ and H(x), requires an additional equation. Especially suitable is an integral condition obtained by a partial integration of eq.(22) over y. This condition reads

$$\boxed{\frac{d}{dx}\left[\int_0^\delta \varrho u (h - h_\delta)\, dy\right] = -\left(\lambda \frac{\partial T}{\partial y}\right)_{y=0}} \; ; \quad \begin{matrix}\delta > \delta_s \\ \delta > \delta_T\end{matrix} \tag{65}$$

The above equation is valid for arbitrary pressure and temperature gradients in the direction of flow [taking eq.(20) into consideration] as well as for arbitrary Prandtl numbers, in both laminar and turbulent boundary layers with /19 variable physical constants. The equation also covers the influence of the heat of friction (dissipation) which is considerable in a compressible flow.

For greater clarity of the solution method, we will restrict the calculation to the case of Pr = 1. If the characteristic quantities of the flow boundary layer, in first approximation, are assumed as known from calculating, in the conventional manner, with $K^{(1)} = 0$ (i.e., with the classical Reynolds analogy), the following differential equation of the first order (using a prime for the derivative to x) is obtained for $K^{(2)}$:

$$\boxed{K^{(2)\prime} + K^{(1)}\varphi_1^{(1)} - \varphi_2^{(1)} = 0.} \tag{66}$$

Here,

$$\varphi_1 = \frac{u_\delta'/u_\delta}{H^* - 1}\left[1 + \frac{\delta_1}{\delta_2} + (\varkappa - 1)M_\delta^2 + H^*(1 - \varkappa M_\delta^2)\right] + \frac{H^{*\prime} + H^{*}\frac{\delta_2'}{\delta_2}}{H^* - 1} \tag{67}$$

$$\varphi_2 = \frac{1}{H^* - 1}\left[b' - b\frac{u_\delta'}{u_\delta}\left(1 + \frac{\delta_1}{\delta_2} + (\varkappa - 1)M_\delta^2\right)\right] \tag{68}$$

with

$$\delta_1 = \int_0^\delta \left(1 - \frac{\varrho u}{\varrho_\delta u_\delta}\right)dy \; ; \quad \delta_2 = \int_0^\delta \frac{\varrho u}{\varrho_\delta u_\delta}\left(1 - \frac{u}{u_\delta}\right)dy \; ; \quad \delta_3 = \int_0^\delta \frac{\varrho u}{\varrho_\delta u_\delta}\left[1 - \left(\frac{u}{u_\delta}\right)^2\right]dy \tag{69, 70, 71}$$

$$\boxed{\frac{\varrho}{\varrho_\delta} = \frac{T_\delta}{T} \quad \text{[because of } \frac{\partial p}{\partial y} = 0 \text{ in eqs.(62)]}} \tag{72}$$

$$\delta_3/\delta_2 \equiv H^*$$

$$\boxed{(H^*)_{\rho/\rho_\delta = 1} \equiv H \quad = \quad \text{form fac[illegible] velocity profile}} \qquad (73)\ (74)$$

The influence of $\frac{dT_w}{dx}$ and $\frac{dp}{dx}$ is expressed in φ_1 and φ_2 by b' [see eq.(60)] and by $\frac{u_\delta'}{u_\delta}$ [see eq.(14)]. The solution for K(x) and for the boundary-layer quantities can be iteratively improved and thus will converge rapidly, according to available data. It should be mentioned that, in this approximation theory, $T_w(x)$ can be prescribed in any form, so that - for example - no expansion in powers of x is necessary. /20

For completeness, we are giving here the integral conditions for momentum and energy in a form found useful for practical calculations of incompressible laminar and turbulent boundary layers.

Instead of the boundary layer thickness $\delta_s(x)$, the following quantity is introduced as parameter for the thickness of the flow boundary layer:

$$Z = \delta_2 R_{\delta_2}^{\,n} \qquad \begin{array}{l} n = 1 \text{ for laminar boundary layer} \\ n = 0.268 \text{ for turbulent boundary layer} \end{array} \qquad (75)$$

with

$$R_{\delta_2} = \frac{\rho_\delta u_\delta \delta_2}{\mu_w} \qquad (76)$$

Then, the "law of the conservation of momentum" reads

$$\boxed{Z' + Z \frac{u_\delta'}{u_\delta} F_1 - n Z \frac{\mu_w'}{\mu_w} - F_2 = 0} \qquad (77)$$

The law of conservation of energy is written in the form of /21

$$\boxed{Z\, H^{*'} + H^{*} Z \frac{u_\delta'}{u_\delta} F_3 - F_4 = 0.} \qquad (78)$$ /21

Here, F_1 to F_4 are universal functions that are fixed by eqs.(68) - (73) with eqs.(57), (58), or (62) that contain the parameters H, M_δ, b (or Θ), and K. Of these, $M_\delta(x)$ and b(x) are known from the problem formulation. Details on this calculus, with more accurate data on the solution arguments for laminar and turbulent boundary layers, as well as the empirical laws for wall shear stress and dissipation at turbulent boundary layer are given by M.Mayer (Bibl.12).

7. Rotation-Symmetrical Boundary Layer

Frequently, heat exchangers are built up of pipe systems with longitudinal flow, on whose inner or outer boundary the rotation-symmetric "start-up" boundary layer reaches a thickness after a certain running length, which is comparable to the pipe radius. In the extrusion of thermoplastic materials in jet nozzles, the threads are exposed as thin cylinders to a longitudinal flow, which stretches and cools them. The boundary layer thickness δ_s may here reach a multiple of the thread radius r_0. Simultaneously, the gradients $\frac{dT_w}{dx}$ and $\frac{dp}{dx}$ may be high.

Here, technically important cases of rotation-symmetrical boundary layers are involved, to which the above equation system for the flow and temperature boundary layers cannot be directly applied.

Under the assumption that the boundary layer thickness δ_s remains small with respect to the running length x $\left(\frac{\delta_s}{x} \ll 1\right)$, the complication relative to the standard theory is relatively minor. However, still another parameter $\frac{\delta_s}{r_0}$ occurs in the equations. At $\frac{\delta_s}{r_0} \rightarrow 0$, the equations (as is necessary) are transformed into the form for normal rotation-symmetrical boundary layers which can be transformed to the two-dimensional case by using the Mangler transformation (Bibl.11).

In his thesis, M.Mayer (Bibl.12) developed integral conditions for momentum and energy from the partial differential equations valid for our case and then worked up all universal functions in these equations for compressible laminar and turbulent boundary layers, based on the improved argument (63) for the temperature boundary layer with the auxiliary unknowns K(x). In the case of a laminar boundary layer, the velocity profiles of Hartree (Bibl.13) were assumed, while "power profiles" were used in turbulent boundary layers. /22

8. Examples

8.1 Two-Dimensional Flow

8.11 Flow along a Plane Plate ($u'_\delta = 0$) with Temperature Gradient

8.111 Laminar Boundary Layer with Constant Physical Constants Pr = 1

This example had been treated by H.Schlichting (Bibl.9). The temperature differences $T_w - T_\delta$ or - using the relations of our approximation theory - the values b of eq.(60) are assumed as being so small that the physical constants can be assumed as $\frac{\rho}{\rho_\delta} = 1$. In that case, the normal flow boundary layer of the plane plate, with Blasius' solution (Bibl.10) for the velocity profile, can be considered as known. Schlichting selected a solution argument for T(x, y) according to powers of the running length $\frac{x}{L}$ with coefficients that are functions of the wall distance y, which has been made dimensionless. For each coefficient, this will yield a linear ordinary differential equation of the second order which, as a rule, can be solved by numerical integration. The series was terminated at the term quadratic in $\frac{x}{L}$. However, this does not in- /23 terfere with the accuracy of the theory and merely restricts the applicability range to temperature distributions $T_w(x)$ up to at most a parabolic character. In the case of a linear distribution of the wall temperature in accordance with the relation

$$\frac{T_w(x)}{T_\delta} = 1 - b(x) = 1 + \left(\frac{T_w(0)}{T_\delta}\right)\left(1 - 2\frac{x}{L}\right) \tag{79}$$

and a Prandtl number of $Pr = 1$, Schlichting obtained the following expression for the local heat flux:

$$\frac{q}{\lambda \frac{T_w(0) - T_\delta}{L} \sqrt{R_L}} = 0.332\left(\frac{x}{L}\right)^{-\frac{1}{2}} - 1.06\left(\frac{x}{L}\right)^{\frac{1}{2}}. \tag{80}$$

Solution of the same problem under the same assumptions and using the above-developed approximation theory, will furnish the following result after a closed integration of eq.(66) with $H = 1.572 = \text{const}$, according to M.Mayer (Bibl.12):

$$\frac{q}{\lambda \frac{T_{w0} - T_\delta}{L} \sqrt{R_L}} = 0.332\left(\frac{x}{L}\right)^{\frac{1}{2}} - 1.153\left(\frac{x}{L}\right)^{\frac{1}{2}}. \tag{81}$$

Both solutions are plotted in Fig.3 against the running length $\frac{x}{L}$. The deviation of the approximation solution from Schlichting's solution is minimal. An interesting point in this result is - as already mentioned by Schlichting (Bibl.9) - that the heat flux $q(x)$ is not zero at the point at which the temperature $T_w - T_\delta$ vanishes, namely, at $\frac{x}{L} = 0.5$, but farther upstream at $\frac{x}{L} = 0.288$ (0.313 according to Schlichting). Figure 4 shows the temperature profiles, calculated from eq.(63), at the points $\frac{x}{L} = 0.288$ and 0.5, which illustrate this statement.

In the case of a parabolic distribution (increase) of the wall temperature, in accordance with the law

/24

$$\frac{T_w(x)}{T_\delta} = 1 + \left(\frac{T_w(0)}{T_\delta} - 1\right)\left(2\frac{x}{L} - \left(\frac{x}{L}\right)^2\right), \tag{82}$$

the results according to Schlichting (Bibl.9) will read

$$\frac{q}{\lambda \frac{T_w(0) - T_\delta}{L} \sqrt{R_L}} = 1.06\left(\frac{x}{L}\right)^{1/2} - 0.635\left(\frac{x}{L}\right)^{3/2}, \tag{83}$$

while, according to the new approximation theory [with a closed integration of eq.(66)], they will read

$$\frac{q}{\lambda \frac{T_w(0) - T_\delta}{L} \sqrt{R_L}} = 1.153\left(\frac{x}{L}\right)^{1/2} - 0.676\left(\frac{x}{L}\right)^{3/2}. \tag{84}$$

Figure 5 shows that the result of these two calculations differs very slightly but that the error with respect to the calculation at constant wall temperature is considerable. For a more detailed discussion of the results, Schlichting's report should be consulted (Bibl.9).

8.112 Laminar Boundary Layer with Variable Physical Constants, Incompressible Flow, $Pr = 1$, $\mu \sim T^{\omega}$, $\omega = 0.7$

The results for the cases mentioned in Section 8.111 are applicable here (as mentioned before) only for such small temperature differences $T_w - T_\delta$ or for values b differing so little from zero that the physical constants can be assumed as being invariant. However, our theory permits a treatment of cases with such large values of the parameter b or of the ratio $\frac{T_w(o)}{T_\delta}$ that a reaction of the temperature boundary layer on the flow boundary layer takes place also in the case of an incompressible flow. However, in this case, the solution is possible only by a numerical integration of the simultaneous system of equations (66), (77), (78). Figure 6, for the example of eq.(79), shows the influence of $T_w(o)$ on $q(x)$. Accordingly, the point $\left(\frac{x}{L}\right)_{q=0}$, within the investigated range of values $1 < \frac{T_w(o)}{T_\delta} < 1.5$, remains practically unchanged at about 0.288. The local heat flux $q(x)$ does not decrease with increasing value of $\frac{T_w(o)}{T_\delta}$. For $\frac{T_w(o)}{T_\delta} < 1.1$, the influence of the variable physical constants /25 is thus negligible.

8.113 Laminar Boundary Layer with Variable Physical Constants, Compressible Flow, $M = 3.0$, $Pr = 0.72$, $\mu \sim T$ $(\omega = 1)$ [Chapman-Rubesin Example (Bibl.6)]

In this classical test example, the system of equations (66) - (78) must be used as basis, without the terms containing the factor u'_δ (because of $u'_\delta = 0$) but under consideration of the compressibility terms (Mach-number terms).

Figure 7 gives the calculation result in comparison with the exact solution. The agreement is satisfactory. In addition, we plotted the computational data obtained according to the approximation theory of Morris-Smith (Bibl.11), which deviates considerably from the exact solution. The curve of the heat flux at constant wall temperature T_w, plotted in Fig.7, indicates the excessive error that might appear when the temperature gradient $\frac{dT_w}{dx}$ is neglected.

8.114 Turbulent Boundary Layer, Invariant Physical Constants, Incompressible Flow, $Pr = 1$

In this example, a closed integration of eq.(66) for $K(x)$ is also possible. Well-known empirical laws apply here to the velocity profile and to the friction coefficient c_f [see, for example, J.Rotta and H.Fernholz (Bibl.16, 17)]. In the case of a linearly varying wall temperature in accordance with eq.(79), which will be the case discussed here, we have

$$\frac{q(x/L)}{\lambda_w \frac{T_w(0)-T_\delta}{L} R_L^{0.847}} = 0.0448\left(\frac{x}{L}\right)^{-0.1535} - \frac{0.0493\,H}{1.668\,H-1}\left(\frac{x}{L}\right)^{0.8465}, \tag{85}$$

(see Fig.8). A vanishing heat flux is obtained here at about $\frac{x}{L} = \frac{1}{3}$, /26
i.e., approximately at the same point as in the laminar boundary layer (see Fig.3). Merely the distribution law for $q\left(\frac{x}{L}\right)$ is different (different powers of $\frac{x}{L}$ and of R_L). In addition, there exists a minor dependence of the result on the form factor H [eq.(74)] of the velocity profile. For $10^5 < R_L < 10^7$, the value of H is located approximately within the range of $1.70 < H < 1.85$.

The absolute values of q, in the case of a turbulent boundary layer, may be several orders of magnitude higher in the mentioned R_L region (because of the higher values c_f) than in the case of a laminar boundary layer, as clearly indicated by a comparison of eq.(85) with eqs.(80) or (81).

The author never has encountered comparative tests for checking these computational data.

8.12 Flows with Pressure Gradient ($u'_\delta \neq 0$) but with Constant Wall Temperature T_w = const, Invariant Physical Constants, Pr = 1, Laminar Boundary Layer

In the examples discussed in Section 8.11, the influence of a wall temperature $T_w(x)$, varying in the direction of flow, on the heat transfer was investigated. It was found that this effect, which had been neglected in the exact solution of Crocco - van Driest (Bibl.4, 5), may be considerable.

We will next have to demonstrate the influence of a pressure gradient in the direction of flow, a point that had also been neglected in the above exact theory. In this case, the wall temperature T_w is kept constant (for separating the influence factors $\frac{dT_w}{dx}$ and $\frac{dp}{dx}$). The temperature differences $T_w - T_\delta$, occurring during the heat transfer, are assumed so small (as in the case of the plane plate in Section 8.111) that the physical constants can be considered invariant and the flow boundary layer as independent of the temperature boundary layer.

For this investigation, the "similar solutions" of the incompressible laminar boundary layer, at potential flows of the type $u \sim x^m$, are especially suitable. /27

According to the exact solution by D.R.Hartree (Bibl.13), the form factor $H = H(m)$ of the velocity profile is constant for each value of the exponent m, i.e., $H' = 0$. For the parameter $Z(x)$ of eq.(75) (for the thickness of the boundary layer), eq.(77) will then yield, because of $H = \text{const}$ and $T_w' = 0$ ($\mu_w' = 0$),

$$\frac{Z(x)}{L} = \left(F_2 - \frac{F_1 F_4}{H F_3}\right)\frac{x}{L} = \text{const}\cdot\frac{x}{L}. \qquad (86)$$

The thickness of the momentum loss will thus become

$$\sqrt{R_L}\,\frac{\delta_2}{L} = \sqrt{F_2 - \frac{F_1 F_4}{H F_3}}\cdot\sqrt{x/L} = \text{const}\cdot\sqrt{x/L}. \qquad (87)$$

With these results, a closed solution of eq.(66) for $K(x)$ becomes possible. According to an elementary calculation, we obtain

$$\frac{K(x)}{b} = -\,m(H)\,\frac{1 + \delta_1/\delta_2}{\frac{H}{2} + m(H)\left(1 + \delta_1/\delta_2\right)}. \qquad (88)$$

In Fig.9, $\frac{K}{b}$ is plotted against $H(m)$.

For the case of a plane plate with $m = 0$, we obtain $K = 0$, as required. In the case of an accelerated flow at $m > 0$, the quantity $\frac{K}{b}$ is negative so that the heat flux, according to eq.(64), will be smaller than in a flow without pressure gradient ($m = 0$). In the case of a decelerated flow (pressure rise), at $m < 0$, the heat transfer is improved over the case of $m = 0$.

In the case of a considerable pressure rise, the variation in heat transfer is considerable. We have scheduled to make a check on this rather surprising and practically significant result, using a difference method and actual measurements.

8.2 Rotation-Symmetrical Flow /28

For checking the above theory, tests on the temperature boundary layer by E.R.G.Eckert, R.Eichhorn, Th.L.Eddy (Bibl.18) are available, who made studies on a cylinder in longitudinal flow $u'_\delta = 0 \left(\frac{dp}{dx} = 0\right)$, with a temperature gradient in the direction of flow. Cases with both laminar and with turbulent flow boundary were investigated. The variation in wall temperature $T_w(x)$ was obtained by suitable electric heating.

Figures 10 and 11 show two distributions of the parameter $b(x) = 1 - \frac{T_w(x)}{T_\delta}$, adjusted to (measured) a laminar boundary layer, together with the distributions $b(x) + K(x)$ calculated on the basis of the theory by M.Mayer (Bibl.12). Figures 12 and 13 give a comparison of the measured temperature boundary layer and of the boundary layer $T(y) - T_\delta$ calculated from eq.(63), for two points x of the cylinder surface. The improvement of the theory, obtained by introduction of the new parameter $K(x)$, was considerable.

A corresponding comparison for turbulent boundary layers has been made in Figs.14 - 17. Obviously, the new theory results in a better agreement with the test data, specifically in the vicinity of the wall (which is decisive for a correct reproduction of the heat transfer).

Finally, Fig.18 shows the result of applying the boundary layer theory to

an example with rotation-symmetric flow and considerable variations in the physical constants, in which the boundary layer thickness δ_s is large compared with the cross-sectional radius r_0 of the body in longitudinal flow (see Sect.7). Here, a smelting jet (scoria), ejected from a platinum nozzle and exposed to longitudinal flow, having a mean temperature of about 1400° C, was involved. The shear forces of a blast jet, flowing at a velocity of about 200 m/sec, attack the surface of this smelting jet and cause its distortion. A combination of these shear stresses with the internal strains of the viscous smelting jet determine the course of the cross-sectional radius $r_0(x)$ in the direction of flow. The calculation of the cross-sectional radius r_0 as a /29
function of the running length x, performed by M.Mayer (Bibl.12), is in satisfactory agreement with the experimental result (according to Fig.18) and thus represents an indirect proof for the usefulness of the above-developed approximation theory.

9. Summary

For solving the problems of heat transfer, as posed by the high-velocity aerodynamics of aeronautics and cosmonautics as well as by modern heat-exchanger and manufacturing technology, the classical theories and computation methods based primarily on Nusselt's work are generally no longer sufficient. The effect of temperature gradients in the direction of flow, which was neglected in these theories, has been proved as appreciable in basic investigations by Chapman-Rubesin (Bibl.6) and Schlichting (Bibl.9) for the case of laminar boundary layers.

An approximation theory is developed for covering the effect of such temperature gradients at arbitrary slope of the wall temperature $T_w(x)$, which is simultaneously applicable to both incompressible and compressible turbulent boundary layers.

The theory also demonstrates (for similar solutions of laminar boundary layers), in an analytically definable form, that the influence of a pressure gradient $\frac{dp}{dx}$ in the direction of flow on the heat transfer, which also had been neglected in the classical theory of heat transfer, might be appreciable.

It is characteristic for this theory that no expansions in series are required for the temperature profile and for the velocity profile of the boundary layer, which would have to be adapted to each example. In the solution argument for the temperature and velocity profile, the basic structure of conventional exact solutions is retained and one free parameter each, for adaptation to /30
general problematics and boundary conditions, is introduced. Thus, the approximation theory, in addition to the boundary layer thickness (momentum loss thickness), contains only a form factor H(x) as the unknown for the velocity profile and a "modification parameter" K(x) as the unknown for the temperature profile.

For determining these three unknowns, a simultaneous system of three ordinary differential equations of the first order is used (integral conditions for momentum, mechanical energies, and total energy). In the two energy equations, the work done by the shear stresses (dissipation) is fully considered. In important special cases, closed solutions of this system are possible (see Sect.8.1). In general, a numerical solution must be performed according to conventional procedures.

Typical calculation examples are given for demonstrating the general applicability and usefulness of the approximation theory.

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FIGURE TITLES

Figs.1 and 2 Functions f_1 and f_2 according to Eqs.(37) - (40) Plotted against the Dimensionless Wall Distance η of the Blasius Solution, for the Flow along a Plane Plate, with the Prandtl Number Pr as Parameter.

Fig.3 Plane Plate in Longitudinal Flow, with Linearly Varying Wall Temperature $\frac{T_w}{T_\delta}\left(\frac{x}{L}\right)$ according to Eq.(79). Assumptions: Laminar Boundary Layer, Invariant Physical Constants ρ, μ, λ, and Pr = 1. Slope of the Local Heat Flux $\frac{q(x)}{\lambda\sqrt{R_L}(T_{w(o)} - T_\delta)/L}$ Plotted against the Running Length $\frac{x}{L}$. Comparison with Schlichting's Result (Bibl.9) Given by Broken Line.

Fig.4 Temperature Profiles $\frac{T}{T_\delta} - 1$ of the Example in Fig.3, at the Points $\frac{x}{L} = 0.288$ and 0.50 for $\frac{T_{w(o)}}{T_\delta} = 1.1$.

Fig.5 Plane Plate in Longitudinal Flow, with Parabolic Distribution of the Wall Temperature $\frac{T_w}{T_\delta}\left(\frac{x}{L}\right)$ according to Eq.(81). Assumptions: Laminar Boundary Layer, Invariant Physical Constants ρ, μ, λ, and Pr = 1. Slope of the Local Heat Flux $\frac{q(x/L)}{\lambda_w \frac{T_{w(o)} - T_\delta}{L}\sqrt{R_L}}$ Plotted against the Running Length. Comparison with Schlichting's Result (Bibl.9) Given by Broken Line. Calculation Result for T_w = const, Dot-Dash Line.

Fig.6 Generalization of the Example in Fig.3: Influence of the Parameter $\frac{T_w(o)}{T_\delta}$ of Eq.(79) on the Local Heat Flux $q(x)$.

Fig.7 Plane Plate in Longitudinal Flow, with Temperature Gradient in the Direction of Flow. Assumptions: Laminar Boundary Layer, Compressible Flow, Mach Number $M_\infty = M_\delta = 3.0$, Pr = 1, $\mu \sim T$. Comparison with Chapman-Rubesin's Exact Solution (Bibl.6) and with Morris-Smith's Approximation Solution (Bibl.11).

Fig.8 Plane Plate in Longitudinal Flow with Linearly Varying Wall Temperature, according to Eq.(79). Assumptions: Turbulent Boundary Layer of $\frac{x}{L} = 0$ up to 1. $Pr = 1$.

Fig.9 Influence of a Pressure Gradient on the Heat Transfer at Constant Wall Temperature, T_w = const. Assumptions: Laminar Boundary Layer, Similar Solutions for $u_\delta(x) \sim x^m$, Invariant Physical Constants, $Pr = 1$.

Figs.10 and 11 Examples of a Wall Temperature Distribution $b(x) = 1 - \frac{T_w(x)}{T_\delta}$ on a Cylinder in Longitudinal Flow, according to Eckert (Bibl.18), and Calculated Distribution of the Function b + K in the Case of a Laminar Boundary Layer.

Figs.12 and 13 Comparison of the Temperature Profiles $T(y) - T_\delta$, Calculated for the Temperature Distributions b(x) of Figs.10 and 11, with Eckert's Experiments (Bibl.18).

Figs.14 and 15 Examples of a Wall Temperature Distribution $b(x) = 1 - \frac{T_w(x)}{T_\delta}$ on a Cylinder in Longitudinal Flow, according to Eckert (Bibl.18), and Calculated Distribution of the Function b + K in the Case of Turbulent Boundary Layer.

Figs.16 and 17 Comparison of the Temperature Profiles $T(y) - T_\delta$, Calculated for the Temperature Distributions b(x) of Figs.14 and 15, with Eckert's Experiments (Bibl.18).

Fig.18 Boundary Layer Theoretical Calculation of the Distortion of a Smelting Jet in a Blast Jet, according to M.Mayer (Bibl.12).

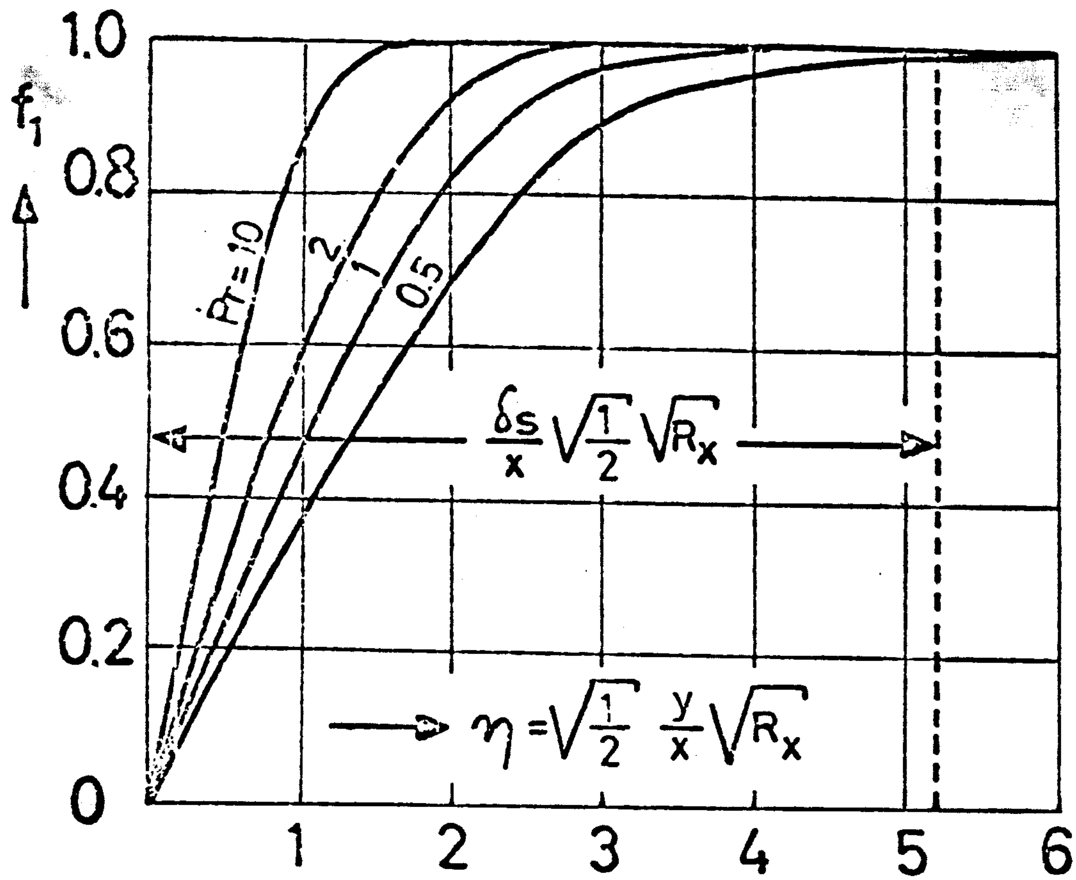

Fig.1 Function f_1 of the Exact Solution according to Crocco - van Driest, for $\frac{\partial T}{\partial x} = 0$, $\frac{dp}{dx} = 0$; Pr = const (Arbitrary)

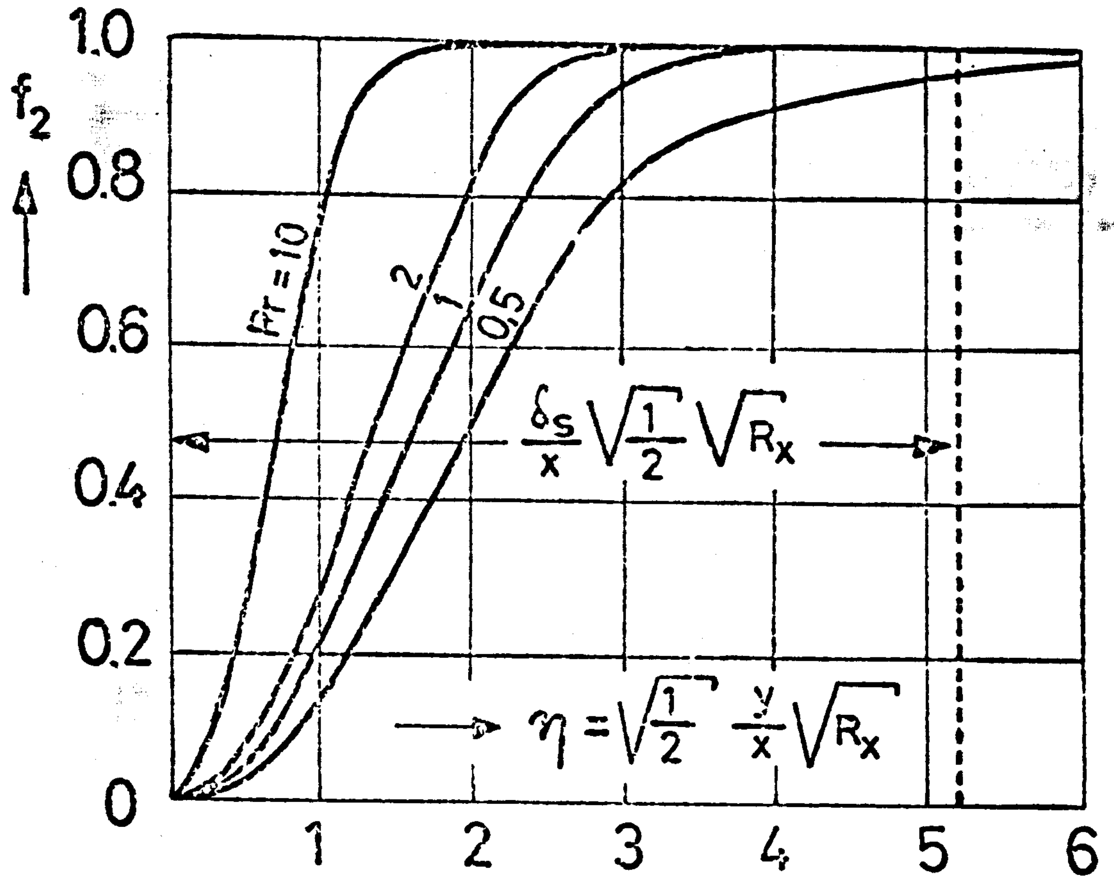

Fig.2 Function f_2 of the Exact Solution according to Crocco - van Driest for $\frac{\partial T}{\partial x} = 0$, $\frac{dp}{dx} = 0$; Pr = const (Arbitrary)

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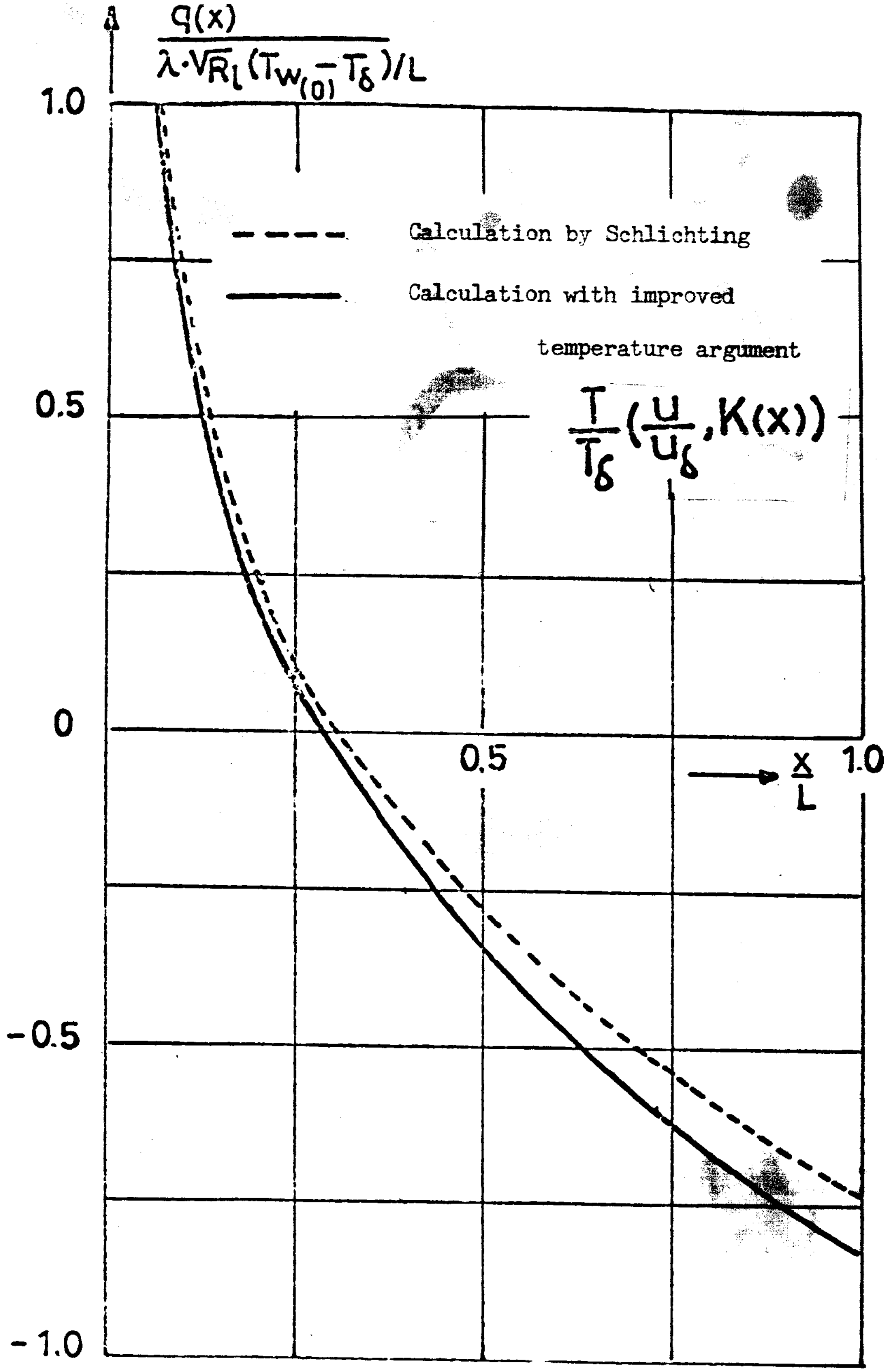

Fig.3 Plane Plate, Laminar Flow, Invariant Physical Constants ρ, μ, λ.
Linear Variation in Wall Temperature T_w

$$T_w(x) - T_\delta = (T_w(o) - T)\ (1 - 2)\ \frac{x}{L}$$

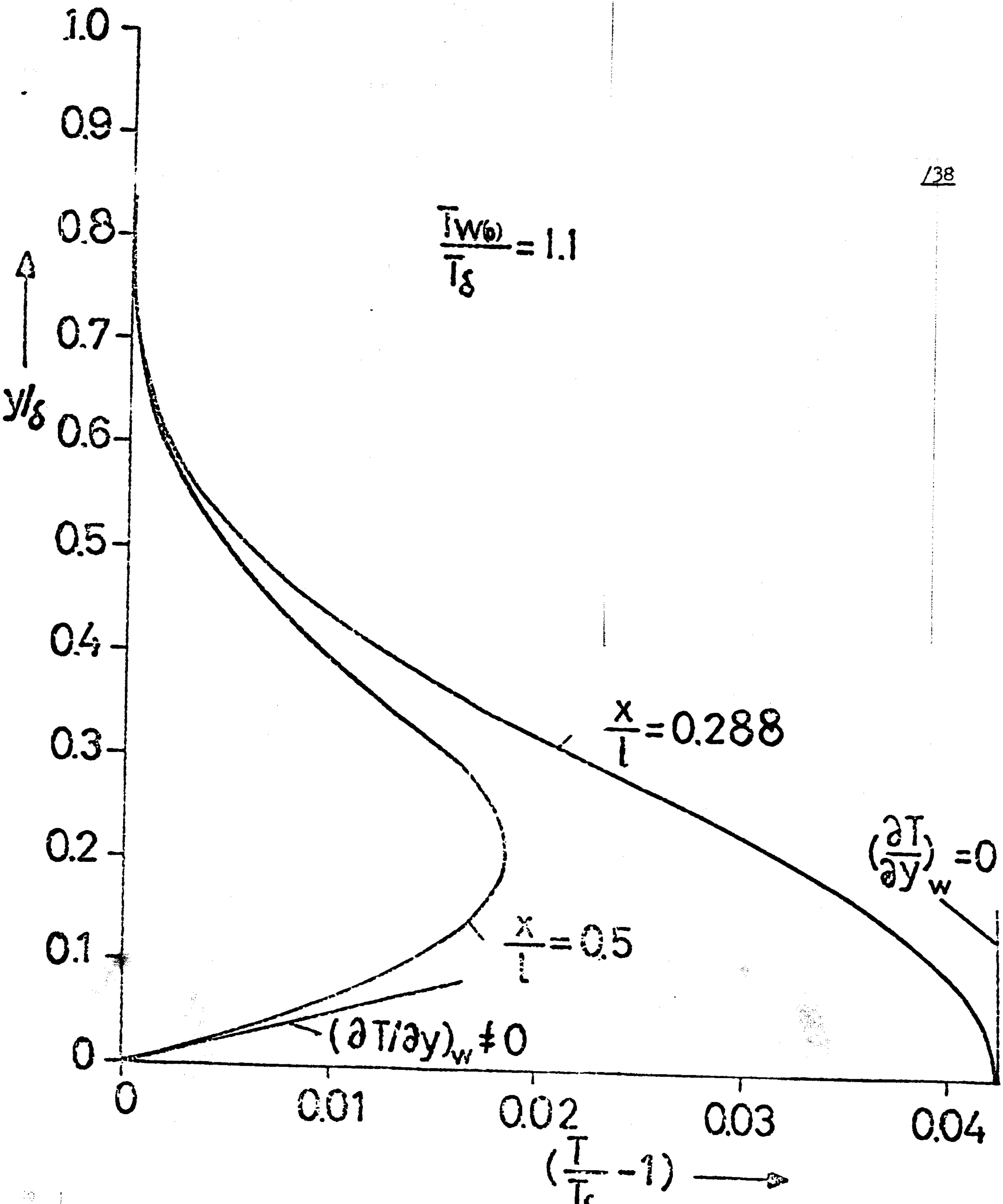

Fig.4 Temperature Profiles $\frac{T}{T_\delta} - 1$ of the Example in Fig.3, at the Points $\frac{x}{L} = 0.288$ and 0.50 for $\frac{T_w(o)}{T_\delta} = 1.1$

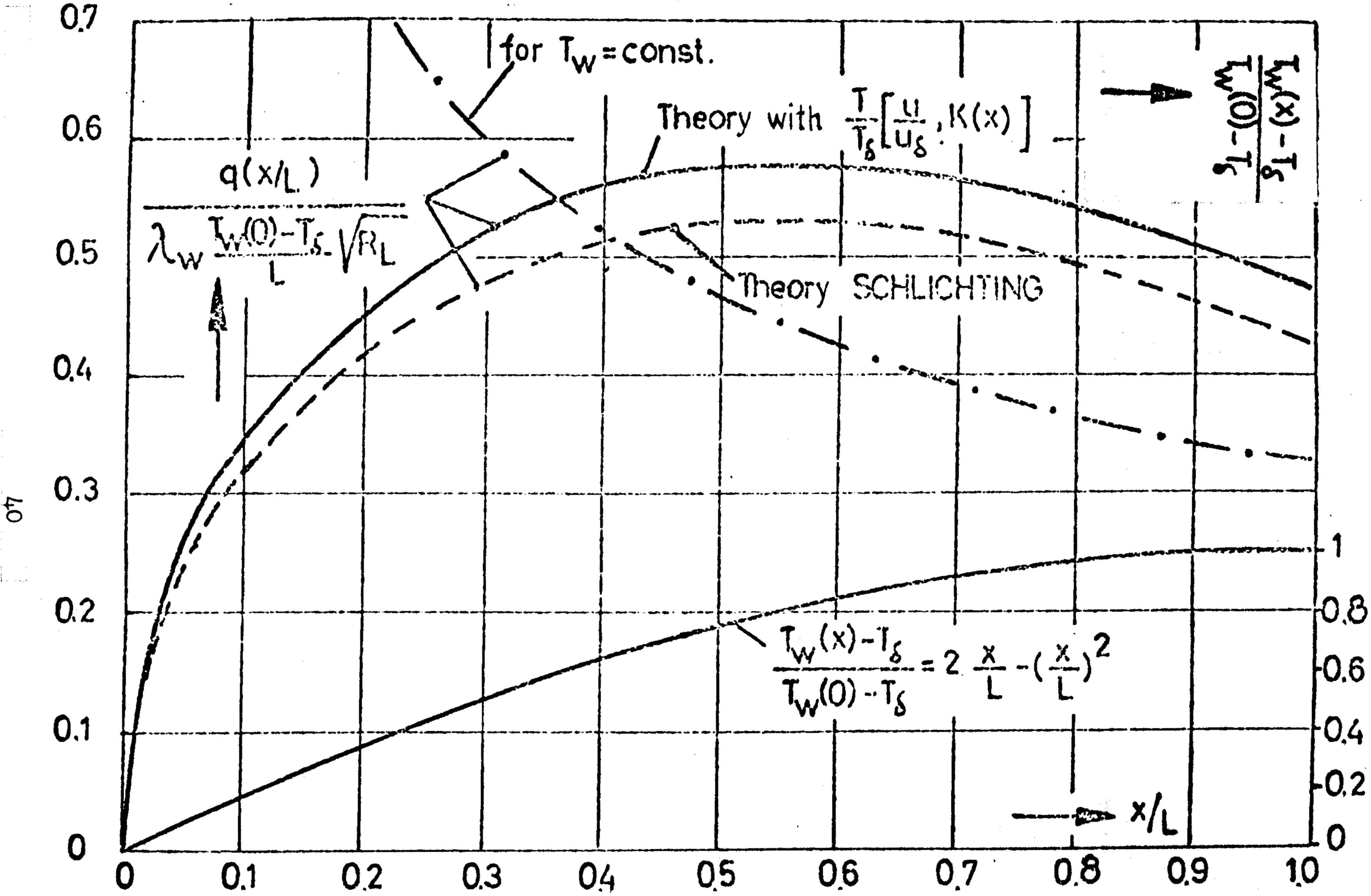

Fig.5 Plane Plate with Parabolically Varying Wall Temperature. Laminar Boundary Layer, Invariant Physical Constants, Pr = 1

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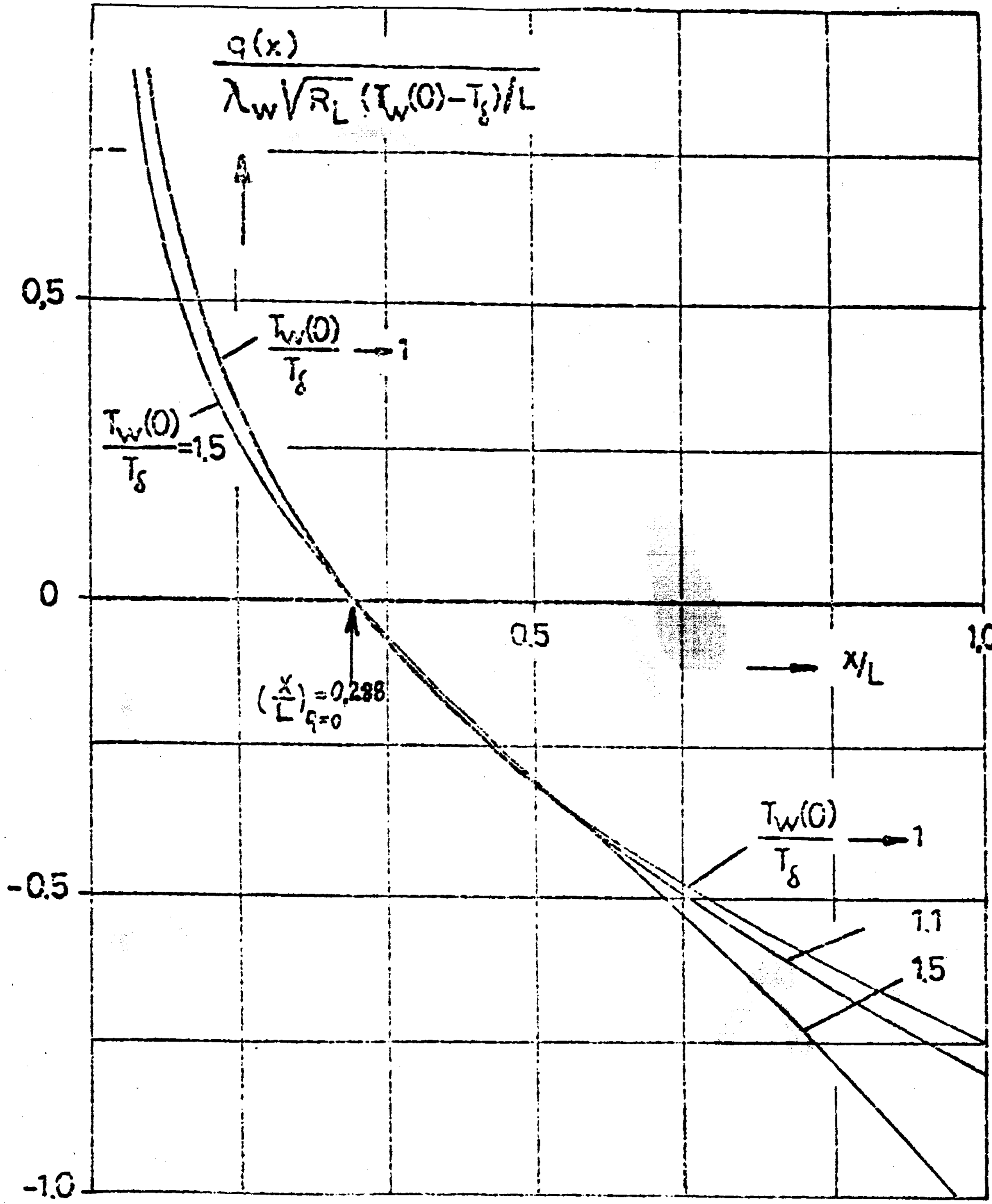

Fig.6 Plane Plate, Laminar, Incompressible Flow, Linear Variation in Wall Temperature. Variable Physical Constants, Influence of $\frac{T_w(o)}{T_\delta}$. Calculation with Modified Temperature Argument $\frac{T}{T_\delta}\left(\frac{u}{U_\delta}, K(x)\right)$

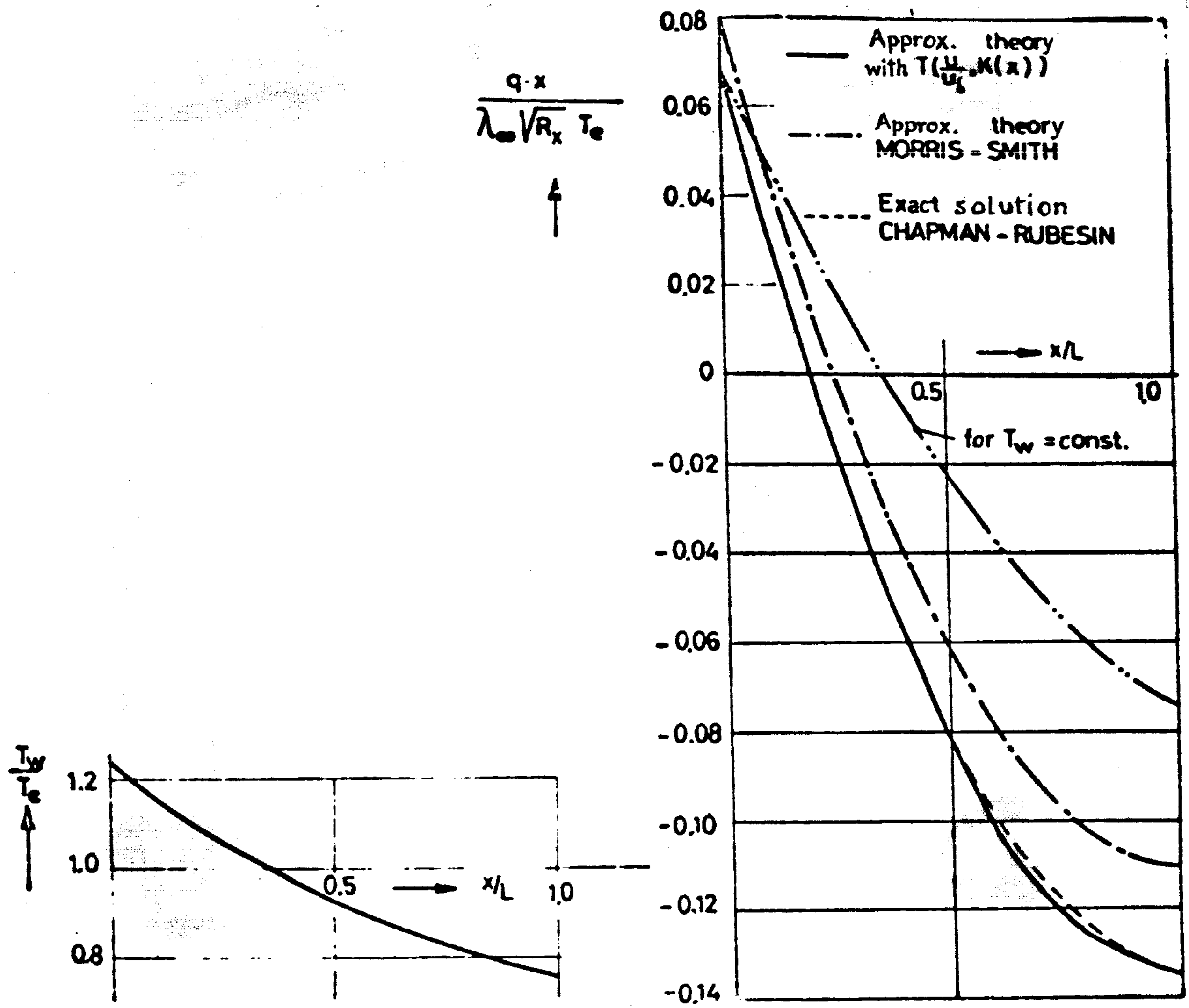

Fig.7 Plane Plate, Laminar, Compressible Flow M_∞ = 3.0, Pr = = 0.72, $\mu \sim T_\omega$, ω = 1, Wall Temperature Slope $\frac{T_w(x)}{T_e}$ = 1.25 - - 0.83 x_L + 0.33 $\left(\frac{x}{L}\right)^2$. Comparison with the Exact Solution by Chapman-Rubesin (Bibl.6) and with the Approximation Solution by Morris-Smith (Bibl.11)

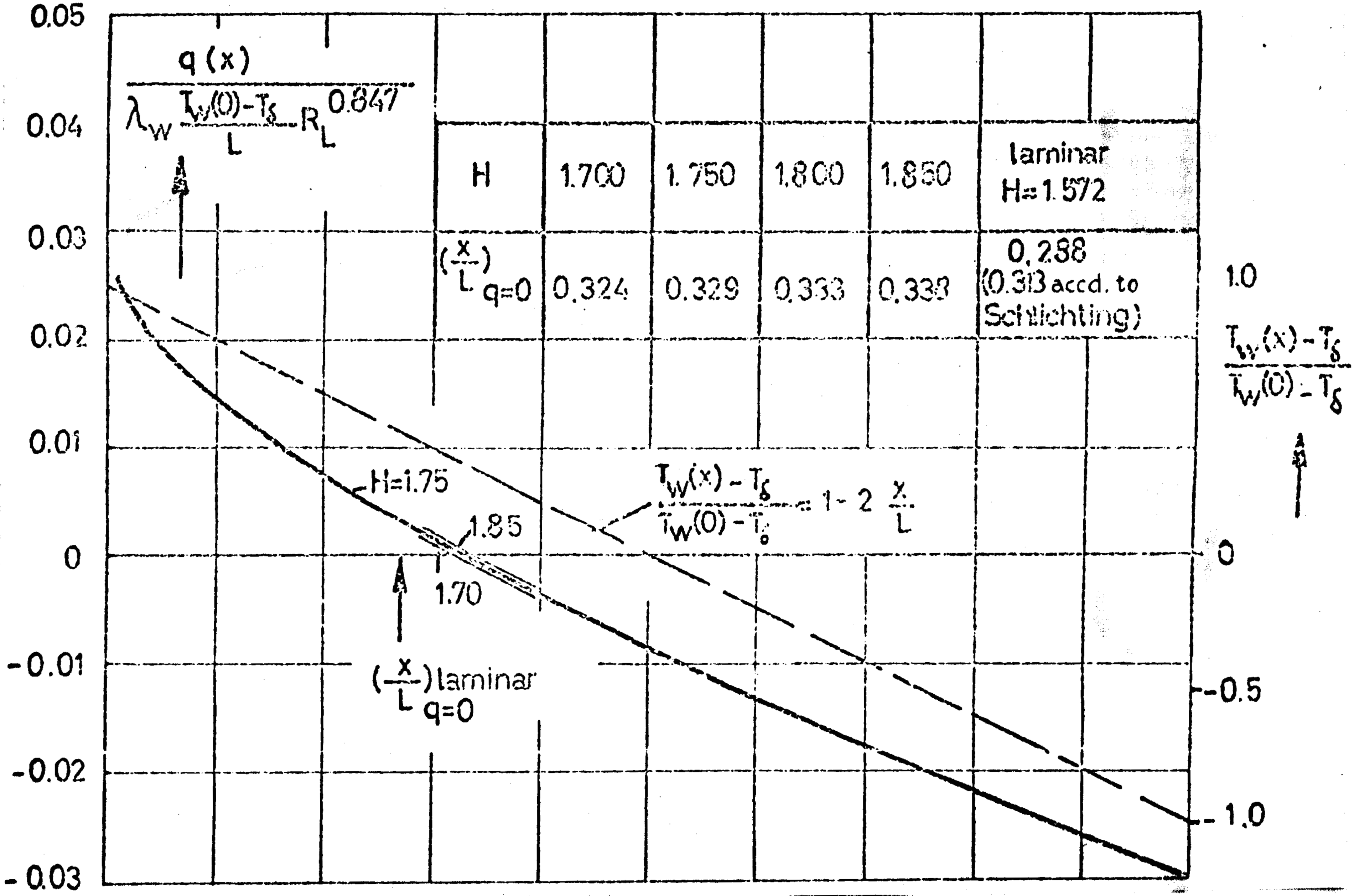

Fig.8 Plane Plate with Linearly Varying Surface Temperature. Turbulent Boundary Layer from x = 0 to 1; Invariant Physical Constants

$$\frac{q}{\varrho_\delta u_\delta^3} = -\frac{c_f}{2}\Theta\left(1+\frac{K}{b}\right) \qquad \Theta = b\,\frac{c_p T_\delta}{u_\delta^2/2}$$

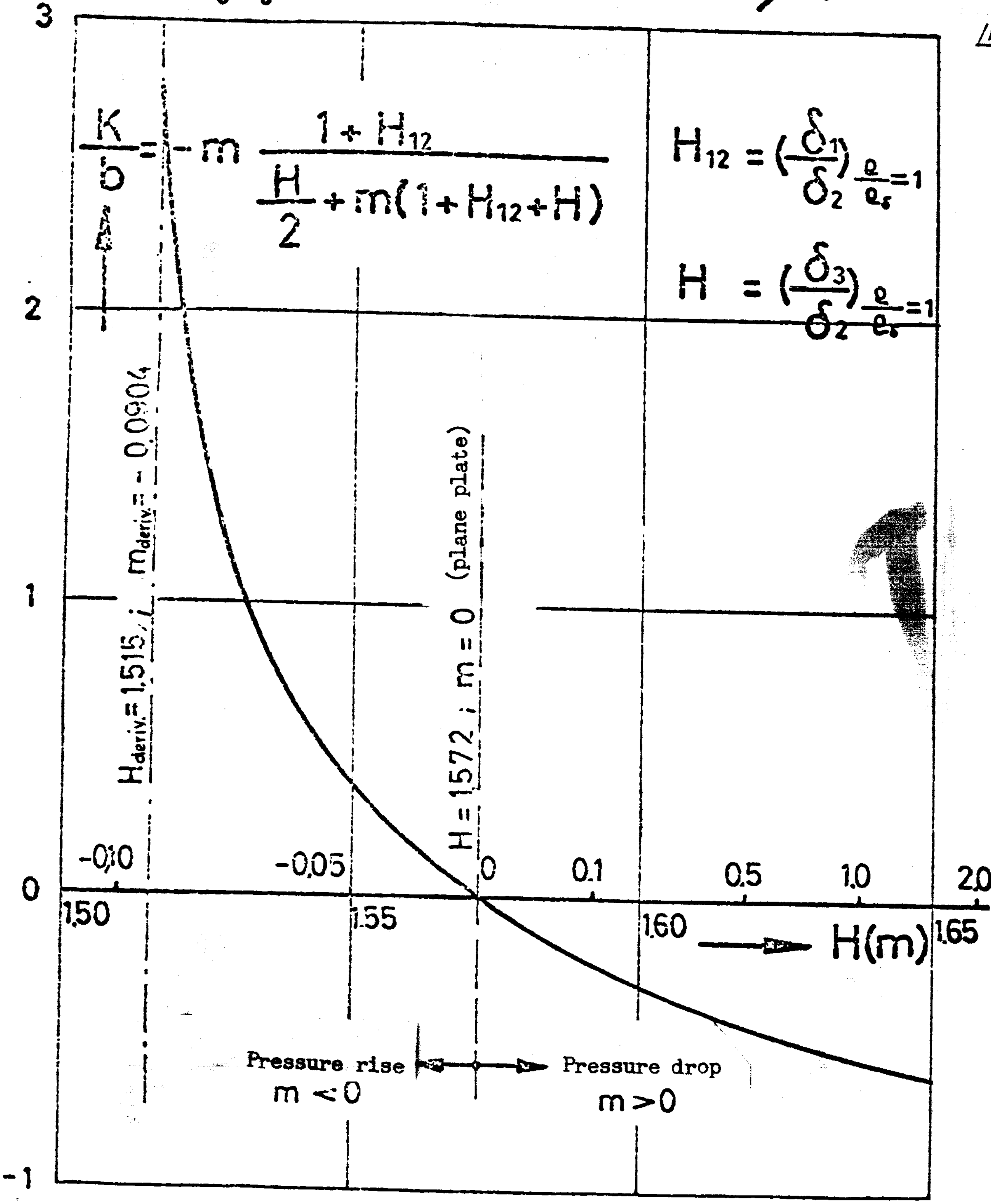

Fig.9 Influence of a Pressure Gradient on the Heat Transfer at T_w = const. Laminar Boundary Layer, $u_2 \sim x^m$, Pr = 1

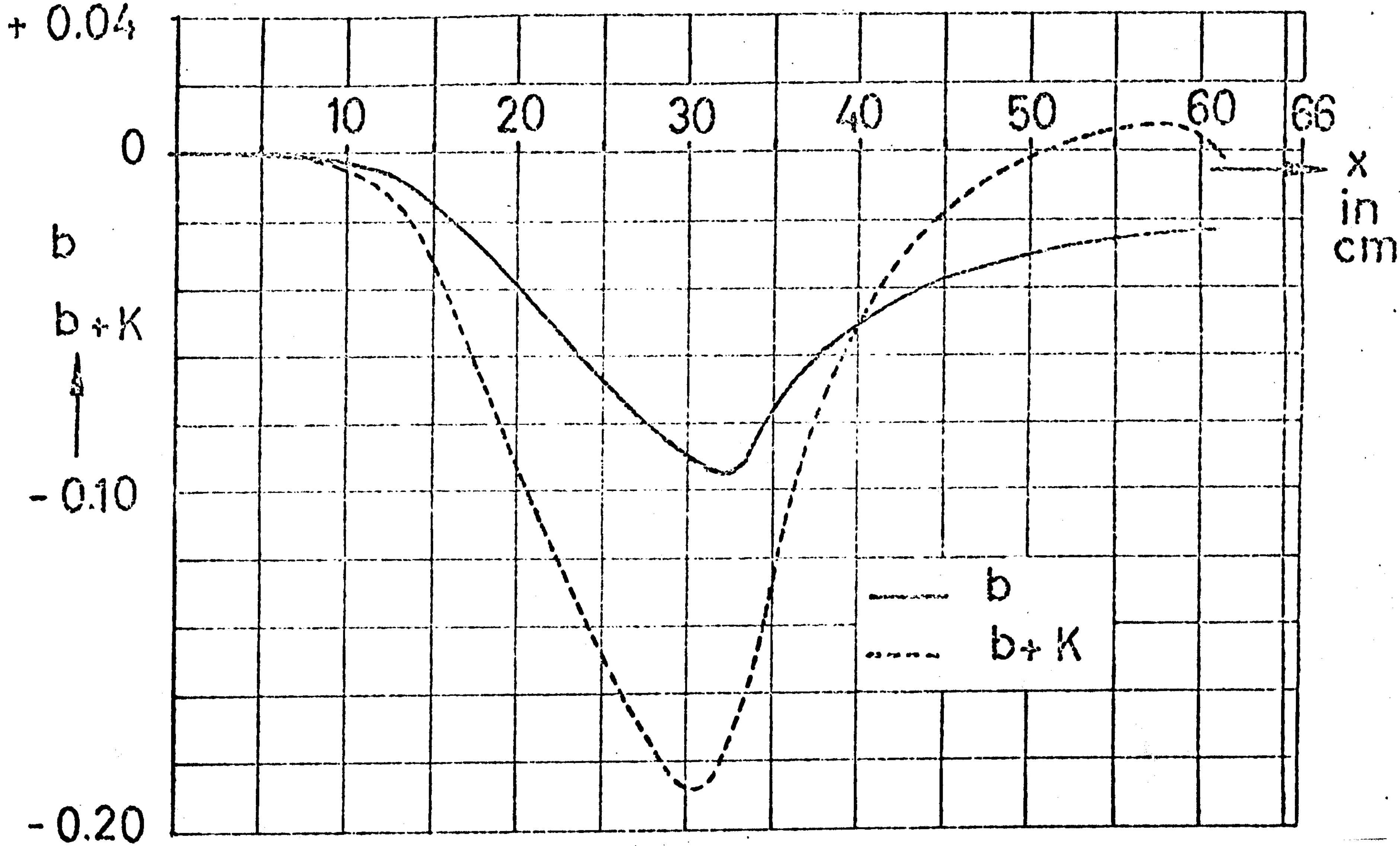

Fig.10 First Example of a Wall Temperature Distribution $b(x) - 1 - \frac{T_w(x)}{T_\delta}$ on a Cylinder in Longitudinal Flow according to Eckert (Bibl.18), and Calculated Distribution of the Function b + K at Laminar Boundary Layer

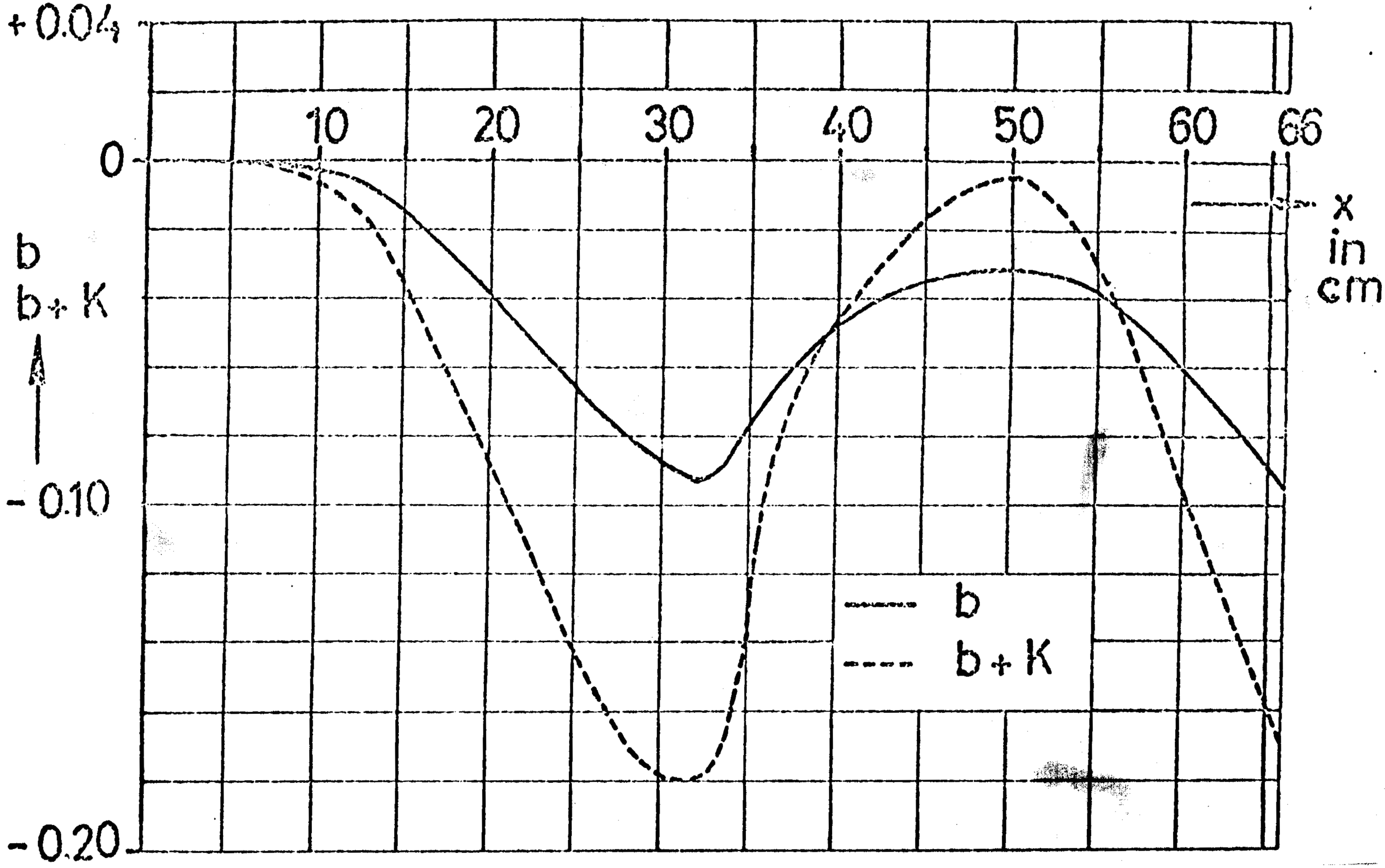

Fig.11 Second Example of a Wall Temperature Distribution $b(x) = 1 - \frac{T_w(x)}{T_\delta}$ on a Cylinder in Longitudinal Flow according to Eckert (Bibl.18), and Calculated Distribution of the Function b + K at Laminar Boundary Layer

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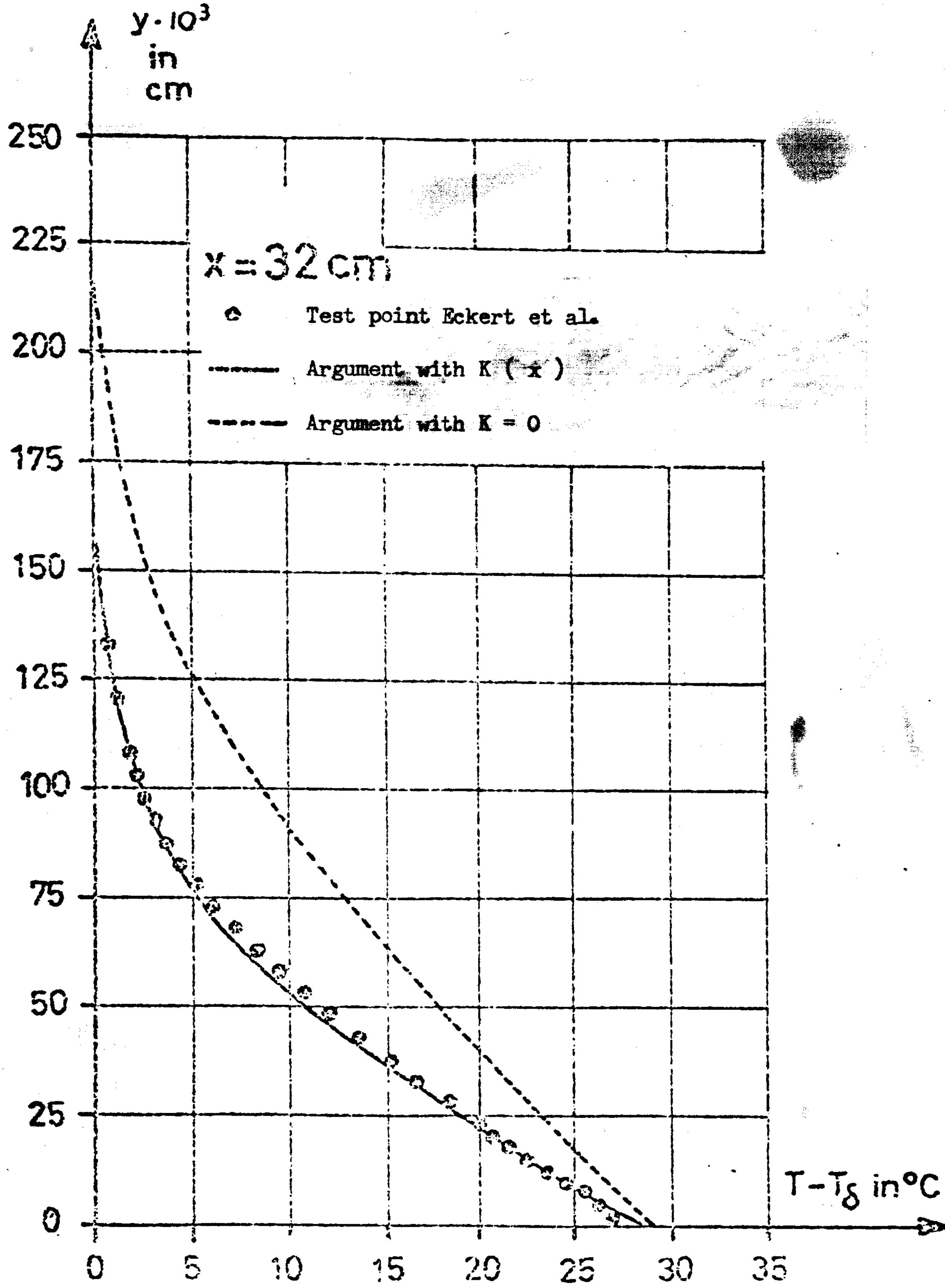

Fig.12 Temperature Profile $T(y) - T_\delta$, Calculated for Fig.10, at Laminar Boundary Layer. Comparison with Eckert's Experiment (Bibl.18)

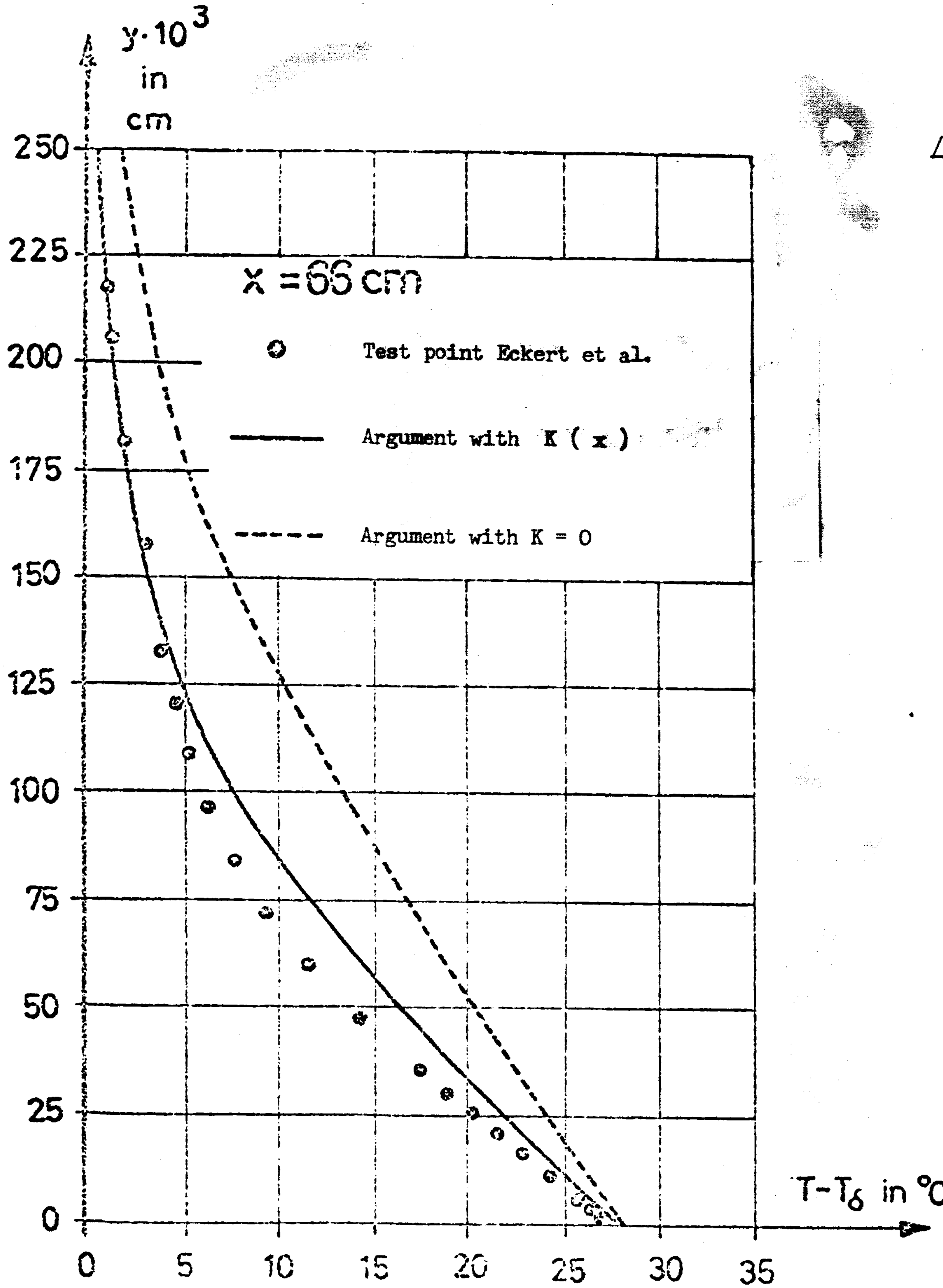

Fig.13 Temperature Profile $T(y) - T_\delta$, Calculated for Fig.11, at Laminar Boundary Layer. Comparison with Eckert's Experiment (Bibl.18)

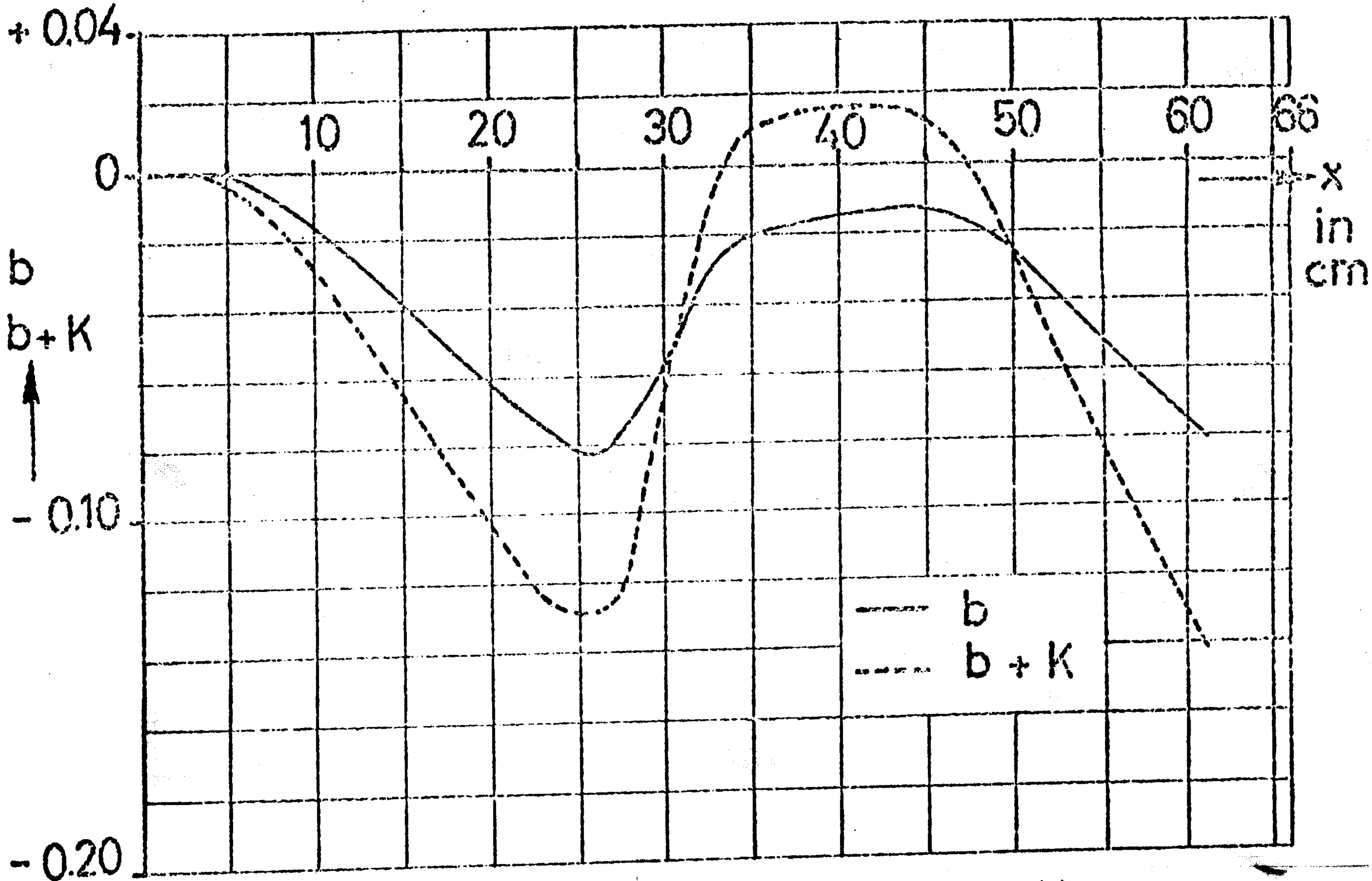

Fig.14 First Example of a Wall Temperature Distribution $b(x) = 1 - \frac{T_w(x)}{T_\delta}$ on a Cylinder in Longitudinal Flow, according to Eckert (Bibl.18), and Calculated Distribution of the Function b + K at Turbulent Boundary Layer

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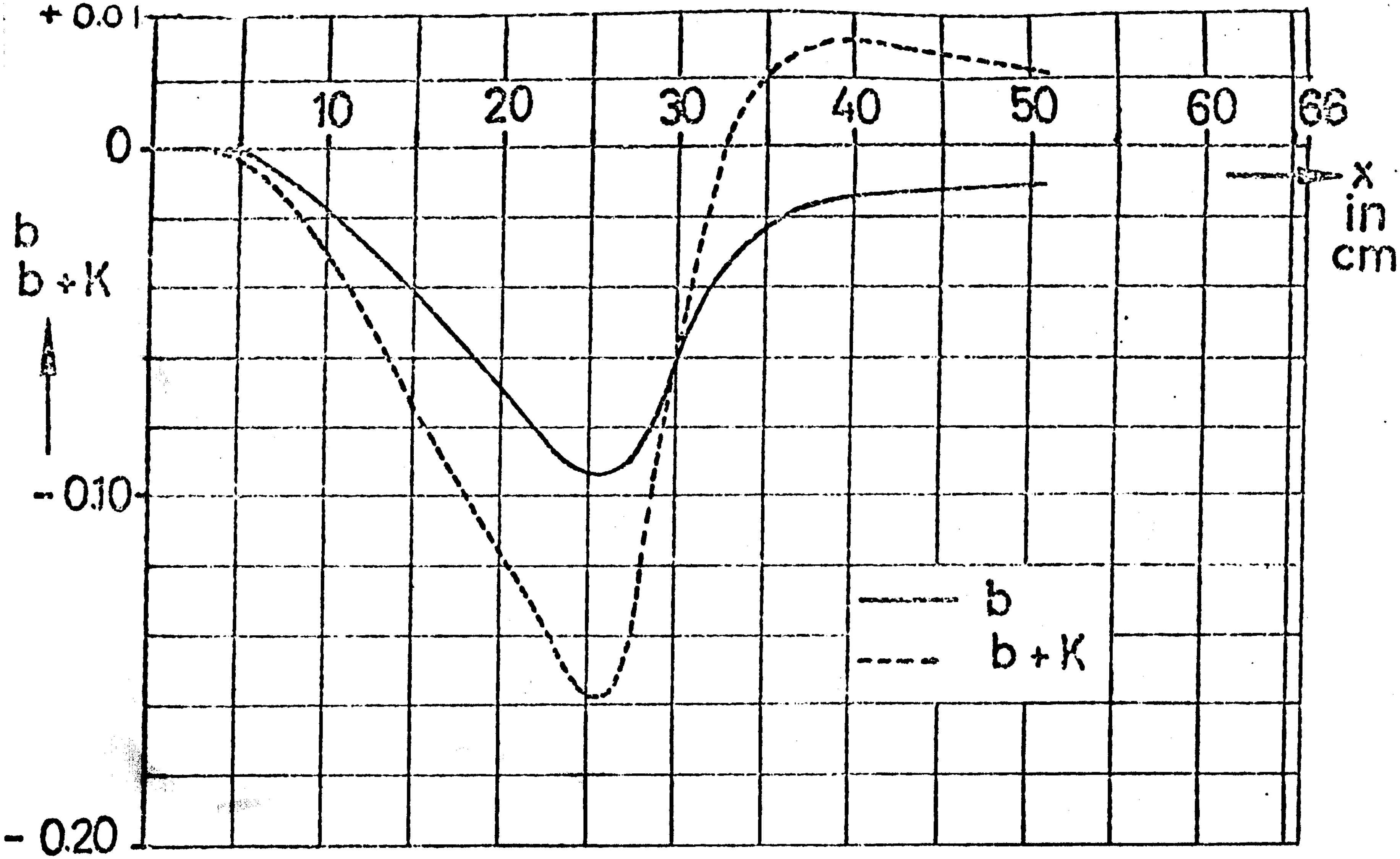

Fig.15 Second Example of a Wall Temperature Distribution $b(x) = 1 - \frac{T_w(x)}{T_\delta}$ on a Cylinder in Longitudinal Flow, according to Eckert (Bibl.18), and Calculated Distribution of the Function b + K at Turbulent Boundary Layer

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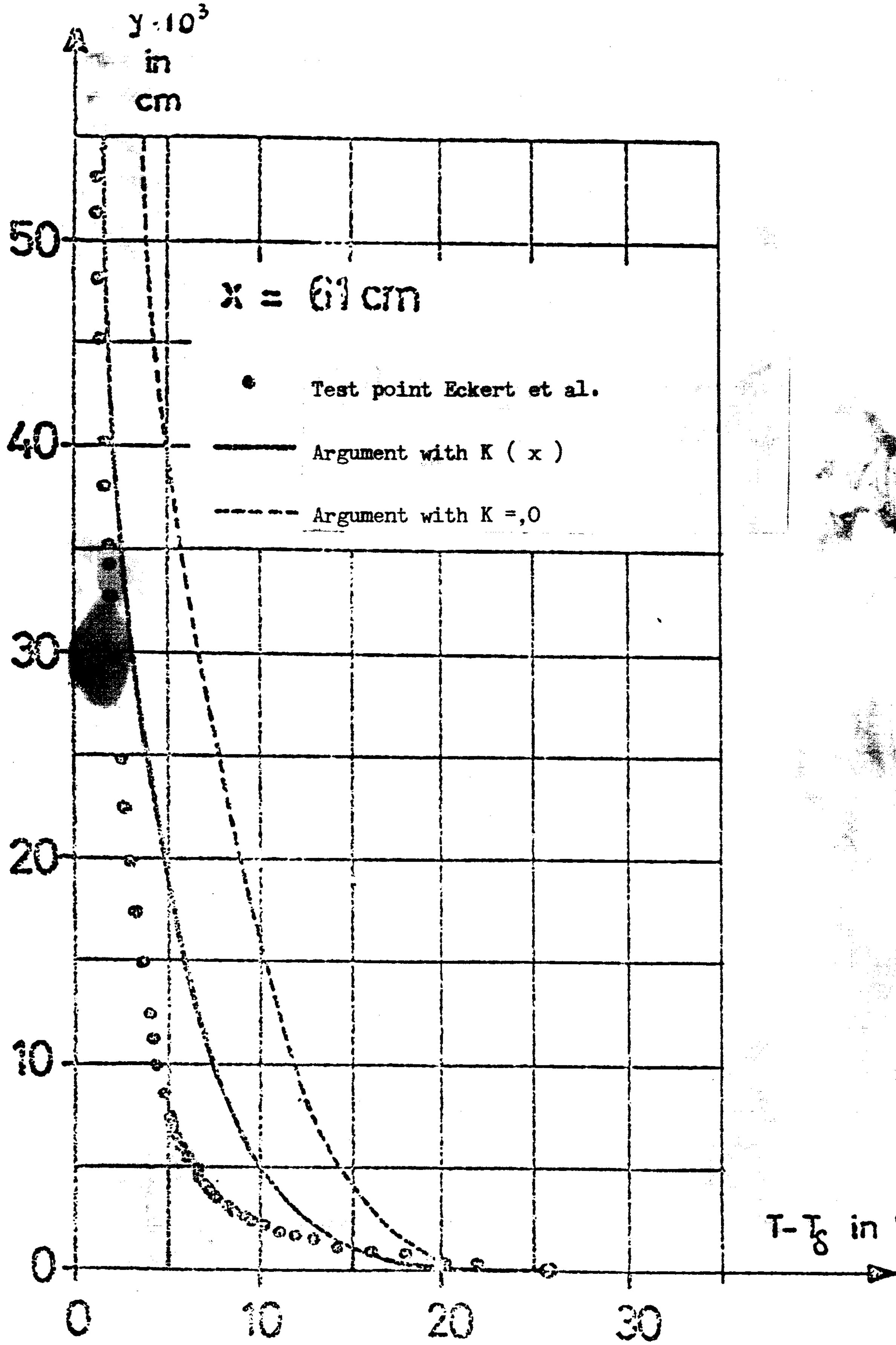

Fig.17 Temperature Profile T(y) - T_δ, Calculated for Fig.15, at Turbulent Boundary Layer. Comparison with Eckert's Experiment (Bibl.18)

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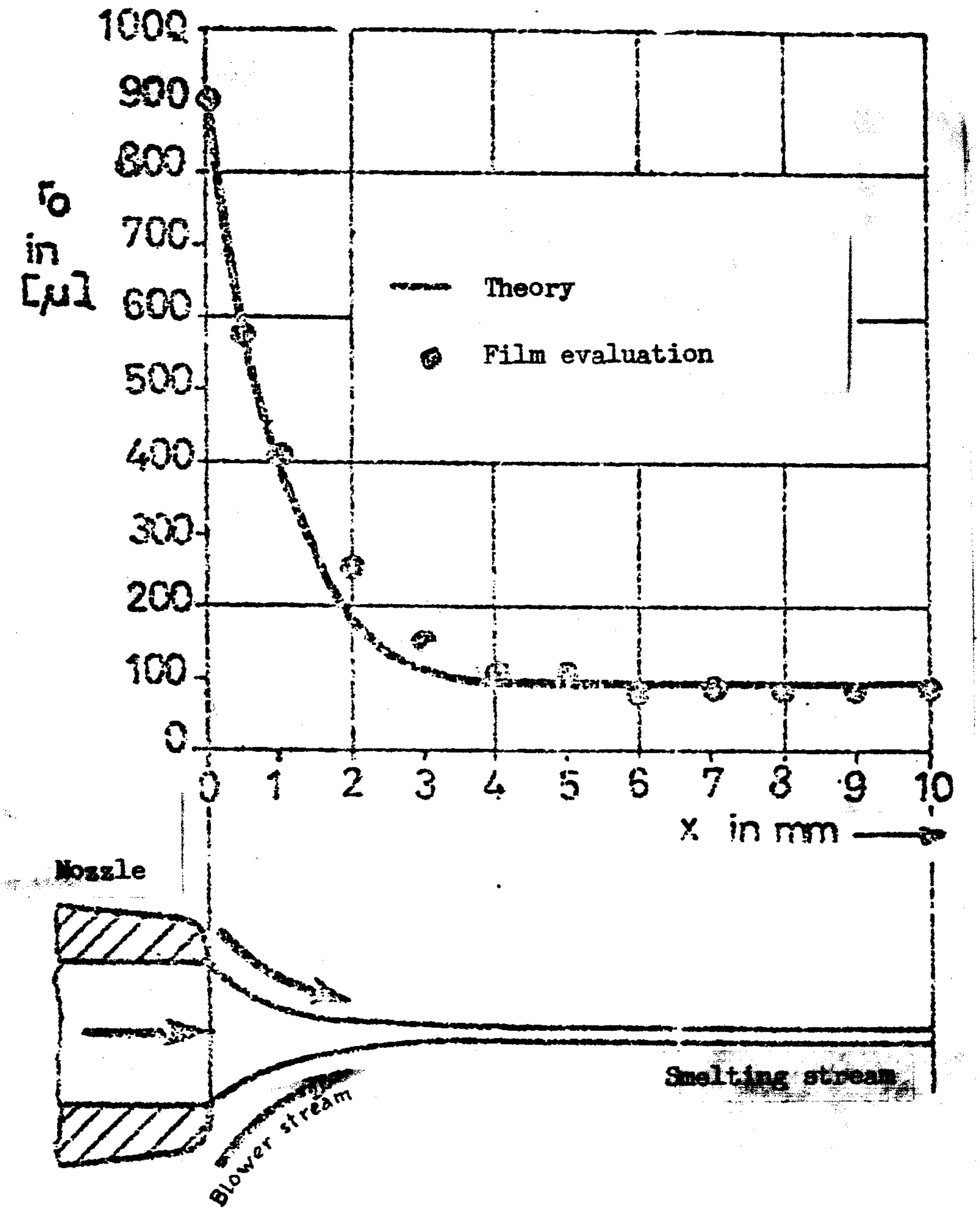

Fig.18 Boundary Layer Theoretical Calculation of the Distortion of a Smelting Jet in a Blast Flux, according to M.Mayer (Bibl.12)